ALGEBRA III
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Let's review what we've learned. I'm going to go slowly. This may seem dry for now but we're going to learn what points are in algebraic geometry.

We learned about the tensor product of algebras, and that $\otimes A$ is a right exact functor. The other thing which is very useful is the following lemma.

What does it mean to extend scalars? This is quite common. You have a ring map $R \to^{\varphi} S$ that makes S an R-algebra. There are two operations people do.

(1) Let M be an S-module. Then M is an R-module via φ (restriction of scalars)
(2) If N is an R-module, how can you make N an S-module? Look to $N \otimes_R S$; this is an S-module.

One is essentially taking a fiber, the other is taking a closure. This second one we've seen, like complexifying a real vector space.

Here's an example; this is the reason why you look to such a lemma as the one I will put. If N is finitely generated as an R-module then $N \otimes_R S$ is a finitely generated S-module with the same generators. This property does not obtain in the extension; there you can have finitely generated modules becoming infinitely generated.

Lemma 1. *$S^{-1}R \otimes_R M \cong S^{-1}M$.*

This duality is sometimes useful. It also says that you don't need $S^{-1}M$, you can just look at the tensor. So how do we prove it? We define two maps.

$S^{-1}R \otimes_R M \to S^{-1}M$
$r/s \otimes m \to rm/s$. This is r-bilinear. You have $r/s \otimes r'm \to rr'm/s \leftarrow rr'/s \otimes m$.

We're writing a grant proposal and the deadline is tomorrow. That's why Lucille came here.

So this is bilinear. Now you have to check that it's bijective. The simplest thing is to produce a map in the opposite direction. We want $\psi : S^{-1}M \to S^{-1}R \otimes M$. So what does this do to a fraction? It is $\psi(m/s) = \frac{1}{s} \otimes m$. You define a map on m and make sure that multiplication with things in s makes things invertible here. Since m goes into $1 \otimes m$, so any multiplication can go over to the fraction.

You also have to make sure it's well defined. It's really elementary. If you have two fractions, you need to go through the proof. You have s'' with $s''(s'm - sm') = 0$. So look at

$$\frac{1}{s} \otimes m = \frac{s''s'}{s''s's} \otimes m = \frac{1}{s''s's} \otimes s''s'm = \frac{1}{s''s's} \otimes s''sm' = \frac{s''s}{s''s's} \otimes m' = \frac{1}{s'} \otimes m'.$$

I'll let you compose these and show that they are the identity.

Corollary 1. *$S^{-1}R$ as an R-module is flat*

If you have $0 \to M' \to M$ of R-modules then you need to show that $0 \to S^{-1}R \otimes_R M' \to S^{-1}R \otimes_R M$.

Exercise 1. *Show that monomorphisms going to monomorphisms is the same as kernels being preserved.*

So I have to show that the map induced in the fractions $S^{-1}M' \to S^{-1}M$ is injective. So you have

$$M' \hookrightarrow M \to S^{-1}M$$

and you want the extension to $S^{-1}M'$ to be injective. So if m' goes to 0 you want $m'/1$ to be 0 in $S^{-1}M$. If m' goes to 0 then $sm' = 0$ so that $m'/1 = 0$, as desired.

So a typical application would have $p \subset R$ a prime ideal. Then $M' \hookrightarrow M$ means $M'_p \hookrightarrow M_p$.

Do this with $\mathbb{Z}$-modules, with your favorite $\mathbb{Z}$-module, your favorite abelian group. This is something you've seen before.

This is related to sheafs in algebraic geometry. So support is going to be the support of a sheaf, the points where the sheafs have nonzero stalk. This is why you see here support. I'm going to say nearby which points are subvarieties.

$Supp(M)$ is the collection of prime ideals such that $M_p \neq 0$.
If M is finitely generated and p is prime then $p \in Supp(M)$ if and only if $p \supset Ann(M)$.

So $S^{-1}M = 0$ if and only if every generator over 1 is 0. So then there is a $s \in S$ which kills each generator. Then the product of these generators annihilates M

If the annihilator is maximal then the support is only one point. So that should rhyme a little with what I've been saying.

Artinian rings look like noetherian, but the condition is reversed. They are the simplest rings you can find in nature.

Proposition 1. *Let R be a ring, M an R-module.*

(1) *If $m \in M$ then $m = 0$ if and only if $m = 0$ in all M_p for p maximal*
(2) *$M = 0$ if and only if $M_p = 0$ for all maximal ideals $p \subset R$.*

So $m = 0$ in M_p means that I can find in the complement of p something complementing m. So $Ann(m) \nsubseteq p$. So $m = 0$ in M means that $Ann(m) = R$. When is an ideal the ring? When it's not contained in any maximal. So $m = 0$ if and only if $Ann(m) = R$ if and only if $Ann(m)$ is not in any maximal ideal if and only if $m = 0$ in all M_p.

So how do you check that $M = 0$? This is true if and only if every element in it is 0, which is equivalent to every element being 0 in every localization.

Corollary 2. *$\varphi : M \to N$ an R-module map is epi/mono/iso if and only if $\varphi_p : M_p \to N_p$ is epi/mono/iso for all p maximal.*

Maximal ideals, please keep in mind, correspond to points. To check that a function is 0, I check that it is 0 at every point. It's very useful, believe me.

Proof. Let's do one of them. φ is mono if and only if ker $\varphi = 0$ if and only if ker $\varphi_p = 0$ for all p. Localization commutes with kernels since it's a flat functor. So this kernel is 0 for all p.

You've just proved the Chinese remainder lemma again. So what does it say? You have a collection Q_i of ideals so that each two are relatively prime. So $Q_i + Q_j = R$. This induces a map from R to $\prod R/Q_i$.

(1) ker $phi = \cap Q_i$. This is obvious.
(2) ϕ is surjective.

Since we've learned this let's prove it in a single stroke. In algebra I you write a formula. Someday you may come up with that formula, some day you might forget it. There's no need.

How to see that it's surjective? Check that localized at every maximal it's surjective. So if m is maximal then at most one Q_i is contained in m.

I hope you trust me that localization commutes with direct sum

Exercise 2. *Show that localization commutes with finite direct sum. It's called "bring everybody to the same denominator."*

So $\phi_m : R_m \to \prod(R/Q_i)_{m/Q_i}$. When is such a thing 0? Now if $Q_i \nsubseteq m$ then $(R/Q_i)_m = 0$. The complements of m are exactly the denominators. So at most one Q_i is in m so either this is $R_m \to 0$ which is obviously surjective, or it goes to the map $R_m \to (R_m/Q_i)_m$, which is again obviously surjective. $[x]$ is the image of x.

So what's next? We look to two properties. One is the ascending chain condition, everything is finitely generated. The other is a dual condition that every chain of ideals stops. One of these is the noetherian rings. The dual property is that every weakly descending chain of ideals stabilizes. These are actually noetherian, and they actually have a structure theorem like the Chinese remainder lemma. We'll end up that the ring is a product of fields. In other words, it's a vector space.

Did you learn the Jordan H older theorem in Algebra I and II? It's like two vector spaces being isomorphic implying that they have the same dimension, but it's for modules. It is about the length of chains.

There is a combination of both that I'm going to write down. It's the structure theorem of Artinian rings. They're the affine coordinate rings of collections of points. So we have the complete description of affine coordinate rings of points.

Now I would like to stop a bit earlier today so I would like to stop here. You'll see this next time. Once I see this thing with points I'd like to go back to the lab. So points are thie simplest things, but they have a lot of geometry.