ALGEBRA III
OCTOBER 12, 2004

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One tool of algebraic geometry is localization; if something is too complicated then localize at a prime and solve it there, and then see what you can recover. We will have A the ring of local functions, like functions defined on a neighborhood, and then M_A modules over A which will correspond to sheaves on that chart. I'm going to discuss this. It requires a level of sophistication to see that charts correspond to collections of primes. Localizing means that you throw out something. We've seen that if you throw out a hypersurface, even starting with something not very nice, you get something affine.

The first theorem will tell me how ideals work between the ring and its localization.

Theorem 1. *(Ideal theory in $S^{-1}R$)*
Let R be a commutative ring, $S \subset R$ a multiplicative system, φ the canonical map $R \to S^{-1}R$.

(1) *$I \subset S^{-1}R \to \varphi^{-1}(I) = J$ is an ideal such that $\varphi^{-1}(I)S^{-1}R = (I^c)^e = I$.*
    As a result, $I \to \varphi^{-1}(I)$ is an injection from the set of ideals of $S^{-1}R$ into the set of ideals of R which preserves inclusions, intersections, and other properties. It takes prime ideals into prime ideals.
    The big question is what is the image?
(2) *$J \in R$ is of the type $\varphi^{-1}(I)$ if and only if $\varphi^{-1}(JS^{-1}R) = (J^e)^c = J$. This is equivalent to the condition that every $s \in S$ is a nonzero divisor modulo J.*

Now let's interpret this. Which are the prime ideals? We'll see a correspondence between prime ideals of $S^{-1}R$ with prime ideals of R which do what? Which have null intersection with S.

There is a third part which says what happens with other operations, which I will write when I've proved this. But this you'll use quite often. If you localized to everything but zero, there are no ideals, no zero divisors, but only the zero ideal fails to meet the multiplicative system, so no surprise.

Subvarieties in the complement of $f = 0$ correspond to ideals in R_f. In another sense it's a density property; if you take the closure you should get back to what you started with, as long as you don't sit in f.

Well, try this at home. These conditions are equivalent.

Okay, now let's prove it.

(1) $I \supset \varphi^{-1}(I)S^{-1}R$. So I have to show the reverse. Take $\frac{r}{s} \in I$. Then $s\frac{r}{s} = \frac{r}{1} \in I$, so that $r \in \varphi^{-1}(I)$. It is the preimage of $\frac{r}{1}$. Now as fractions, this generates $r \cdot \frac{1}{s}$ so $\frac{r}{s}$ is now in $\varphi^{-1}(I)S^{-1}R$.

I hope that you don't want me to show that this preserves inclusion, intersections, and so on. Why does it take primes to primes?

$R/\varphi^{-1}(I) \hookrightarrow S^{-1}R/I$. So if I is prime then the range here is a domain, so then the domain is a domain, so then the preimage is prime.

So let's do the second one. Don't say this is too simple, you'll see it is so useful in just a little while.

(2) Let $J = \varphi^{-1}(I)$. Then $JS^{-1}R \subset I$. From the first part the preimage of this is J.

Look to $S^{-1}R/I$. So elements of S act on this ring as nonzero divisors, since they are invertible. Now $S^{-1}R/I$ contains $R/\varphi^{-1}(I)$. So if elements of S act as nonzero divisors on the big ring, they do so on the smaller ring as well.

Now we just have to show that if this is the case, then J is a restriction. Let me think of what I said, just one second.

So if I want J to be a preimage I look to what happens when it is for my candidate, which turns out to be $JS^{-1}R$. So I want to show that J is the preimage of this. So J is contained, but I have to show that $r \in \varphi^{-1}JS^{-1}R$ sits in J.

The whole thing is translation of definitions. This means $\frac{r}{1}$ sits in $JS^{-1}R$. Then $\frac{r}{1}$ is $\frac{j}{u}$. That means there exists $s' \in S$ with $(ru - j)s' = 0$. So $s'ru = s'j$. Here $s' \in S$ and $j \in J$ so $s'j \in J$. Also $s'u \in S$. So something multiplies r into J, then I can deduce $r \in J$, which is what I want.

The zero divisor way is somehow the most useful way to say this.

Let's see applications. This is a basic tool of the trade.

Corollary 1. *Say R is noetherian. Then $S^{-1}R$ is Noetherian.*

How do we show this? I'm hoping that we're going to use this machine. Take an ideal to show it is finitely generated. Look to $\varphi^{-1}(I)$. Then $I = \varphi^{-1}(I)S^{-1}R$. So the generators of $\varphi^{-1}(I)$ generate I over $S^{-1}R$. That's the whole proof.

Okay, now let me make one more comment. If you have an ideal on one side you can extend to the other; on the other you can contract to the first.

So if I have an ideal like this, an extensible ideal, I can go from $I \subset R$ to $S^{-1}I$, which is shorthand for $IS^{-1}R$.

Corollary 2. *This S^{-1} commutes with*

- *finite sums $S^{-1}(I+J) = S^{-1}I + S^{-1}J$.*
- *products*
- *intersections*
- *radicals $S^{-1}(\sqrt{I}) = \sqrt{S^{-1}I}$ (this is the only interesting exercise here)*
- *for the colon we have $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$ as long as I is finitely generated.*

Exercise 1. *Show all of these except the colon.*

How do we find maximal and prime ideals? With Zorn's lemma.

Proposition 1. *Let R be a ring, $S \subset R$ a multiplicative system, and take I an ideal maximal with respect to nonintersection with S. Then I is prime.*

Take $f, g \notin I$, with $fg \in I$. There are two ways of proving this. Let's do it in several ways, let's amuse ourselves. Say $f, g \notin I$. Then $I \subsetneq I + (f), I \subsetneq I + (g)$. Then $I + (f), I + (g)$ meet S. Then you can find two elements $i + af \in S$ and $j + bg \in S$. Here $i, j \in I$. So then we try to make their product. Say $fg \in I$. Now all of the terms are in I so that the whole thing is in I and something in I meets S.

Now let's prove this Zeng's way. Extend I to the fractions. Then the ideal $IS^{-1}R$ is maximal; it's contained in a maximal ideal whose preimage is prime and contains I but does not meet S. So the extension of I is maximal, and then its preimage contains I but does not meet S, so it is equal to I. Then $I = \varphi^{-1}(IS^{-1}R)$ so I is prime.

Now let's see one quick application of this.

Corollary 3. *The radical of an ideal is the intersection of all primes containing I.*

We've shown this, but let's show it again.This is equivalent to saying that the nilradical is the intersection of all prime ideals. If something is a radical in R then it's in the nilradical of R/I. Then use the correspondence between ideals in the quotient and ideals of R containing I.

So it's obvious that every nilpotent lies in every prime ideal, since some power of it is $0 \in p$. Say $f \in R$ and f is not nilpotent. Then I can produce a multiplicative system. I look now for an ideal which is maximal with respect to this multiplicative system. It's prime and does not contain f. So f is not in this intersection.

So you can see that this is very powerful.

Now I want to change gears a little bit. I need to make a detour for tensor products. How many have seen the tensor product of two modules over a ring. These correspond in algebraic geometry to fiber products. I'm going to say what is a fiber product. Let's do it in two steps.

Say R is a ring, commutative, A, B R-algebra. Then I can take $A \otimes_R B$ an R-module. But this is actually an R-algebra. What is the product. You do it componentwise, so that $(a \otimes b)(c \otimes d) = (ac \otimes bd)$. You have to make sure that this is a well-defined thing. Is this bilinear on each side? It is, because you have distributivity.

I should warn you that topologists have a slightly different look at this. Their algebras are not commutative, they're skew-commutative. If $A = \oplus A_i,\ R = \oplus R_i$, i.e., these are graded, these can be skew-commutative. For most people this is the same as commutative once you learn a rule. If $r \in R_i, y \in R_j$, then $xy = (-1)^{ij} yx$. The same thing works for A, so that it's a skew commutative R-algebra. So if you have two, you want $A \otimes B$ to be skew-commutative.

What operation in algebraic topology corresponds to this? If A is the cohomology of one space, B the other, then the tensor product is the cohomology of the product. This is the Kunneth formula. So how do you make this skew-commutative? You define $(a \otimes b)(c \otimes d) = (-1)^{bc} ac \otimes bd$. The magic trick is that the power of -1 is the product of the terms that have to jump over each other.

This has also a universality property, as you'd imagine. Given C an R-algebra and two morphisms $\alpha : A \to C, \beta : B \to C$, then there exists a unique extension $\alpha \otimes \beta : A \otimes_R B \to C$. I'm going to be interested in having a multiplication. You need a thing from $A \times B$ which is bilinear. So you go from (a, b) to $\alpha(a)\beta(b)$. This is bilinear so it passes through, and conversely.

This may seem cryptic. Imagine that these are affine coordinate rings for different algebraic sets. Then these morphisms correspond to pullbacks, which turn out to be polynomials.

Let's look to a simpler situation. Let A be an R-algebra. Let R be the affine coordinate ring for X, and A be the affine coordinate ring for Y. Then this gives a polynomial from Y to X.

So the fiber product is exactly the collection of all pairs in the cartesian product which map to the same thing to make the following diagram commute.

$$\begin{tikzcd} Y \arrow[r] & X \\ Y \times_X Z \arrow[u] \arrow[r] & Z \arrow[u] \end{tikzcd}$$

There is another operation with tensor products that I'm going to use quite often. It's a good idea to look up some of this. The only thing is that you don't actually use the definition of the tensor product. You only use the universality properties and the property that $\otimes_R$ is right exact. Say I have a short exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0$$

and N is an R-module, then I can tensor to get a sequence

$$M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

So it may not be injective but it keeps cokernels. So what comes at the beginning? The tor functor will.

The other thing is that there are modules which make this flat.

Definition 1. *If $\otimes_R N$ maps short exact sequences to short exact sequences, then N is called a flat R-module.*

So modules which preserve short exact sequences are called flat. Why are they called flat, why not tensor? I want to do this, but I can't do it today, it may take another two months. This has to do with proper submersions in the language of algebraic geometry.

One example of a flat module is the ring itself. Free R-modules are flat, because you just replace it with a number of such copies. This notion is credited to Grothiendieck and others.

The main thing about localization is that it gives a flat R-module. Flatness means you preserve monomorphisms and epimorphisms. It preserves kernels, cokernels, everything.

There are two statements and I'll end.

(1) $S^{-1}R \otimes_R M \cong S^{-1}M$
(2) $S^{-1}R$ is flat.

These are the two important things with the tensor product. This is the only time you're interested in going backward. These are the two facts I'm going to state quickly. This follows from the fact that the short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

induces a short exact sequence

$$0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$$

So this and the first of these give the second.

Next time maybe applications, probably.