ALGEBRA III
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So, today I want to essentially say a couple of words about Artinian rings, but from a more applied point of view. It will require a little bit of topology; if you don't know the topology then take it on faith.

In what follows, k will be a field, most of the time $\mathbb{C}$ and sometimes $\mathbb{R}$.

One way to get a local ring is to localize, at a prime ideal (subvariety) or a hypersurface defined by a function.

There are other rings which become local, essentially by fiat.

Proposition 1. *$k[[x_1, \ldots, x_n]]$ (any field), $k\{x_1, \ldots, x_n\}$ (convergent power series over $\mathbb{R}$ or $\mathbb{C}$, i.e., $\mathscr{O}_n$), $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ are local rings*

The second one here is exactly germs of holomorphic functions. If you put together all the germs you get a sheaf. So this is a local manifestation of the sheaf; it's a natural ring to look at. This is a local ring, as opposed to $\mathscr{O}(U)$.

You also have an inclusion $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} \subset k\{x_1, \ldots, x_n\} \subset k[[x_1, \ldots, x_n]$. The first of these is in some sense the algebraic-geometric localization, the second the analytic localization, the third the formal localization.

The unique maximal ideal is $(x_1, \ldots, x_n)$ in all of these. Say $g \notin (x_1, \ldots, x_n)$; Then up to a constant this is $g = 1 + h, h \in (x_1, \ldots, x_n)$. So the inverse of $1 + h$ is usually $\sum(-h)^i$. When h a formal power series has no constant term, this sum does not involve infinitely many terms in any variable, so if g is a formal power series not in $(x_1, \ldots, x_n)$ then g is a unit.

The same is true for the ring $k\{x_1, \ldots, x_n\}$; the inverse converges for a small enough disk around the origin.

Finally, in the smallest ring, you have something like $\frac{f}{g} = f(\sum(-h)^i)$ which shows that you have an inclusion. No one cares about the formal power series but the other two are very important. The second one contains more information because it includes functions which are not polynomials.

What are the typical things in which we are interested? Take an ideal in the polynomial ring which vanishes at the origin; you localize to find the multiplicity of its vanishing.

Let's try an example, then I'll put definitions.

Let $I = (x^2 + x^3, y^2) \subset \mathbb{C}[x, y]$.
This is the zero locus of two polynomials. Let $A = k[x, y]/I, \dim_k A = lt(A) = 6$.

$V(I) = \{(-1,0),(0,0)\} \subset \mathbb{C}^2$. In particular this ideal is not radical, and shows you that the vanishing happens with multiplicity.

How do I calculate the multiplicity of these points in such a way that the multiplicities add to the dimension? This is how things work in one variable.

If $I \subset k[x_1, \ldots, x_n], \#V(I) < \infty$ and $0 \in V(I)$ then

$$mult_I(0) = \dim_k k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}/Ik[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} =$$
$$\dim_k k\{x_1, \ldots, x_n\}/Ik\{x_1, \ldots, x_n\} = \dim_k k[[x_1, \ldots, x_n]]/Ik[[x_1, \ldots, x_n]].$$

The last two equalities are lemmas. To do it off the origin, you localize at the ideal of that point.

The computation of this particular example is very easy. I won't prove it, but I will try to give evidence for a result of Milnor in '68 that is useful for exotic spheres among other things.

We have $mult_0(I) = \dim_k k[x,y]_{(x,y)}/(x^3+x^2,y^2)k[x,y]_{(x,y)} = k\{x,y\}/(x^2+x^3,y^2)k\{x,y\}$. I'm claiming these are equal.

So $x^2(x+1)$ is the first generator, but $x+1$ is invertible with inverse $1-x+x^2+\cdots$. So the dimension is the dimension of $\dim_k k[x,y]_{(x,y)}/(x^2,y^2)k[x,y]_{(x,y)}$.

What is the inverse of $1+h$? Look to h^3? Any polynomial is divisible by either x^2 or y^2 so is zero in this ring. Since this is a local ring, the Gr obner basis will transfer from one ring to another, which proves the lemmas. The basis for these is $1, x, y, xy$ so we have multiplicity four.

So let's compute for the other zero; let's do it by changing coordinates. Instead of localizing at $x+1, y$ call these u, v and localize $k[u,v]_{(u,v)}/((u-1)^2+(u-1)^3, v^2)k[u,v]_{(u,v)}$.
The polynomial in u is $u^3-2u^2+u = u(u^2-2u+1)$. Since u^2-2u+1 is invertible you get the ideal (u,v^2) in the local ring and $(1,v)$ is the basis. So the multiplicity here is two and we get six total, as we should.

Theorem 1. *Let $I \subset k[x_1, \ldots, x_n], k = \bar{k}, \#V(I) = \{p_1, \ldots, p_m\} < \infty$. Then*

$$\dim_k(k[x_1, \ldots, x_n]/I) = \sum_{i=1}^m \dim_k(\mathscr{O}_i/I\mathscr{O}_i) = \sum m_I(p_i).$$

Here $\mathscr{O}_i$ is one of the three localizations at p_i.

We proved this, actually. Let me convince you. The quotient is an Artinian ring; it's the direct sum or product of local rings, since a finite length module is a direct sum of localizations at maximal ideals. The maximal ideals in such a quotient are the maximal ideals of the polynomial ring which contain I; then these are precisely the points. What we proved was that the quotient was $\prod \mathscr{O}_i/I\mathscr{O}_i$.

Corollary 1. *$I = \sqrt{I}$ if and only if $m_I(p_i) = 1$ for all $p_i \in V(I)$.*

That is because the dimension of the quotient $k[x_1, \ldots, x_n]/I$ is at least the dimension of the quotient by the radical. But for a radical ideal the dimension is the number of points so for the sum of the multiplicities to be equal to the sum of copies of the number one, it is necessary that every multiplicity be one.

If you just want to do convergence, you have to make sure that the things you invert are convergent, if you want to do your computation in the convergent ring.

The books will talk about analytic algebras, which are precisely $\mathscr{O}_i/I\mathscr{O}_i$.

The initial terms in some order will be the smallest power terms. All of the Gr obner basis stuff works in these rings, but with different kinds of orders. The bases are the same, morally, but somehow different.

Macaulay cannot compute in local rings but another program, Singular, does. I'll show you maybe next time we go there.

Why are we interested in multiplicities? Say I have a germ of a holomorphic function $f \in \mathscr{O}_{n+1}$. I look at the germ $(X, 0) = (\{f = 0\}, 0)$. Say $f = y^2 - x^3 \in k\{\{x, y\}\}$. Then $(\{f = 0\}, 0)$ is not a submanifold; it is a cusp, since the partials vanish here. I don't need to localize for $k[x, y]_{(x,y)}/(3x^2, 2y)k[x, y]_{(x,y)}$ since its variety is the single point 0.

I'm looking at $\mu(f) = k[x_0, \ldots, x_n]_{(x_0,\ldots,x_n)}/J_f k[x_0, \ldots, x_n]_{(x_0,\ldots,x_n)}$ which only gives you something meaningful for J_f singular.

If ϵ is small and I look to B_ϵ around 0 in affine space, then its boundary is a $2n + 1$ dimensional sphere of radius ϵ.

If f is a polynomial I can see f as a function $\mathbb{C}^{n+1} \to \mathbb{C}$.

There is some finite set in $\mathbb{C}$ such that if you look over the complement, then $f : \mathbb{C}^{n+1}\backslash f^{-1}(B_f) \to \mathbb{C} - B_f$ is a locally trivial fibration. The idea is that if you remove the origin, all of the fibers will look the same.

What Milnor proved was that $S^{2n+1}_\epsilon \cap \{f = 0\}$ is a $2n - 1$ dimensional submanifold. If $n = 1$ then this is a collection of circles, i.e., a link.

If f is irreducible in $\mathscr{O}_{n+1}$ (This is a UFD, which I didn't prove; it is different than polynomial irreducibility, an example is $y^2 = x^2(x + 1)$. At the origin this factorizes as $(y - x)(y + x)$.) then f is a knot, not a link. The number of factors in the convergent power series is the number of components of the link. These are examples of fibered links. The intersection of the ball is the cone over your link.

If in the sphere you get rid of your link then you've got a map $S^{2n+1}_\epsilon \cap X = K_\epsilon$, a link. Then I can talk about the complement (you have to do things a little different if you're working over germs of holomorphic functions), where f takes nonzero value. Then you can map to the sphere by $x \to \frac{f(x)}{|f(x)|}$. Then this looks like a bouquet of spheres, with a number of spheres equal to $\mu(f)$.

Take a branch for one part of a link; take a branch for the other; the multiplicity of their intersection is the linking number. So there is a very nice interpretation of this. It's a story that can be read from a very nice text from '68 or '69. It has this theorem, that the number of circles in the bouquet is equal to $\mu(f)$.

So I'm in S^3, the boundary of the ball, and for ease I'll take $\epsilon = 1$. Where would you do a stereographic projection? You want $(1 - t) + ta_0 = 0$ which gives you $t = \frac{1}{1-a_0}$. So the projection gives $(a_0, \ldots, a_n) \to (\frac{a_1}{1-a_0}, \ldots, \frac{a_n}{1-a_0})$. So with two variables this goes $(x, y, z, t) \to (\frac{y}{1-x}, \frac{z}{1-x}, \frac{t}{1-x}) = (u, v, w)$.

Do a note. Look to $f = xy$. This has two branches. I have two lines $V(f) = \{x = 0\} \cup \{y = 0\}$. If I intersect one line with the sphere I get a circle $\{u = 0, v^2 + w^2 = 1\}$.

For the second line I get the same thing; $\{u \in \mathbb{R}, v = w = 0\} \cup \{\infty\}$. This is a Hopf link. Look to the complement. This fibers over S^1. I'll let you decide what the surfaces are. I claim that the fibers are one sheeted hyperboloids, which shrink to a circle. The other example I did, which is a lot more interesting, was to take $f = x^2 - y^3 = 0$; you can parameterize this by $x = t^3, y = t^2$. So $(x, y) \in S^3$ if $|t|^6 + |t|^4 = 1$, so you get $(\lambda^3 e^{3i\theta}, \lambda^2 e^{2i\theta})$, the $(3, 2)$ torus knot, the trefoil.

This is topology more than algebra, but it is topology that can be computed very nicely by algebra.