ALGEBRA III
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GABRIEL C. DRUMMOND-COLE

Primary decomposition continues. The plan for today is to finish the definition and tell you what you get. Let's review some notions. I don't remember if I proved, but R was a Noetherian ring, M a finitely generated module, and I made two definitions.

We said $N \subset M$ is p-primary if $Ass(M/N) = \{p\}$. We say that M is coprime if (0) is p-primary, i.e., if $Ass(M) = \{p\}$.

We discussed the proposition:

Proposition 1. *The following are equivalent*

1. *M is p-coprimary*
2. *$p \supset ann(M)$ is minimal and every element outside p is a nonzero divisor on M.*
3. *$p^N M = 0$ for some $N \geq 0$ and every element outside p is a nonzero divisor on M.*

This means $I \subset R$ is primary if and only if every zero divisor in R/I is nilpotent, if and only if $xy \in I, x \notin I$ implies $y^n \in I$ for some n. This implies $p = \sqrt{I}$ is prime. The converse is false except for maximal $\sqrt{I}$.

I'm going to sketch this a little bit. To see if an ideal is primary you calculate this last condition.

Let's prove one implies two.
$Ass(M) = \{p\}$, so that $p \supset ann(M)$. I claim that this is minimal with this property. You have to remember what we learned. If you look to primes minimal over the annihilator they are contained in the associator. So if this is minimal there is another prime in between in the associator which is not possible.

Then the zero divisors of M including zero is the union of $p \in Ass(M)p = p$.

So two implies three. It's enough to prove after I localize at p. This means that R is a local ring with maximal ideal p.

I still know that p contains the annihilator. Things outside p don't multiply things to zero. We know that p is minimal over $ann(M)$, but it's maximal, so contains every other prime containing the annihilator. Then the radical of the annihilator is p itself. Then some power of p is in $ann(M)$.

Now I prove that three implies one. So I know p is nilpotent modulo the annihilator. This means that p is minimal and contains $ann(M)$. Why is that? Any other prime containing the annihilator contains a power of p and thus p.

But a prime minimal over the annihilator is in the associator. Can I have anything else? Try to use exactly what I used on the other board. I have to prove equality. Remember, the primes in the associator are those primes which are annihilators of elements. But the collection of zero divisors is contained in p. So all the associated primes are contained in p, so a zero divisor of M is contained in p so $\{p\}$ is the associator of M.

I've been using a few of these pieces of machinery a lot. I'll come back to this. There is a lot of geometry behind this.

Let's show primary decomposition. I'm going to use this. What does it say, primary decomposition? It says that what you got through irreducible decomposition is already good.

Theorem 1. *Lasker-Noether*
Let R be a Noetherian ring, M a finitely generated R-module, nonzero. Then any proper submodule N is an intersection of finitely many primary submodules.

That roughly says that $N = \cap_{i=1}^{t} M_i'$, where M_i' is p_i-primary and p_i are prime ideals.

(1) *Every associated prime of M/N occurs among the p_i.*
(2) *If the intersection is irredundant, then the collection of the p_i coincides with the associated primes.*
(3) *If moreover the intersection is minimal, (t is minimal) then each associated prime of M/N is isomorphic to one of the p_i for exactly one i.*
(4) *Primary decomposition localizes; if you take a multiplicative system, then the primary decomposition of $S^{-1}M$ consists of only the M_i' for which there is no intersection with S.*

I also want to make a remark on the proposition which I didn't make. How do I interpret that M is p-coprimary? Look to $M \to M_p$. This is injective because the complement of p has no nonzero divisors. This gives an equivalent condition if we include also that $p \supset ann(M)$ is minimal.

If M is a module and $p \supset ann(M)$ is a minimal prime, then I can go from $M \to M_p$. I have a kernel $0 \to M' \to M \to M/P$. You deduce that $M/M' \hookleftarrow M/P$. But $M_p = (M/M')_p$. So what does it say if p is a minimal prime? It says that M/M' is p-coprimary. In other words, M' is p-primary.

For minimal primes over the primary, there is an M' which is p-primary to show up in the intersection. This is going to give some uniqueness. M' is called a p-primary component of M, and those are the geometry components.

Let's try to sketch a proof; note that there is no uniqueness in general, there are only certain kinds of uniqueness.

This is what you would think of as taking subsets, but it doesn't always work. So let's prove this.

We took $N = \cap M_i$, for these irreducible. We showed that the a.c.c. showed this was true. The next claim is that if M_i is irreducible, then it is primary.

What do I have to show?

Lemma 1. *If $M' \subset M$ is irreducible then M/M' is coprimary.*

Say it is not, so $p, q \in Ass(M/M')$. Then p, q annihilate elements. So $R/p \hookrightarrow M/M'$. The same is true for R/q. Then the intersection of their images is 0. Any element of R/p has annihilator p; an element of R/q has annihilator q. So for an element to be an image of both it has to be annihilated by both, so it's 0. But now I'm in trouble, because then I get $M' = \tilde{L_1} \cap \tilde{L_2}$.

Now let's show quickly one two three.

How do we show the first thing? An observation: These things have nothing to do with N, we can divide by N and assume N is (0). Minimality and redundancy commutes with quotients. So $(0) = \cap M_i$. I can send M to $\oplus M/M_i$. The kernel is 0 because it's the intersection. Then the associator of M is included in $\cup Ass(M/M_i)$. So every associator of M is in the set of p_i.

How about part two? What does it mean irredundant? It means that $\cap_{j \neq i} M_j \neq (0)$ for all i. Then $M_i \cap (\cap_{j \neq i} M_j) = 0$ so $(\cap_{j \neq i} M_j)/(M_i \cap (\cap_{j \neq i} M_j))$ is $(\cap_{j \neq i} M_j)$, but this is the same as quotienting $\cap_{j \neq i} M_j + M_i$ by M_i. These are submodules of M so the quotient is a submodule of M/M_i. But this last is p_i-coprimary.

The intersection $\cap_{i \neq j} M_j$ is nonzero and its associator is contained in $Ass(M/M_i) = \{p_j\}$ so we get equality.

For part three I leave for you as an exercise.

Lemma 2. *If you have two M_1, M_2 p-primary then $M_1 \cap M_2$ is p-primary.*

This is because $M/M_1 \cap M_2 \hookrightarrow M/M_1 \oplus M/M_2$. So $Ass(M/M_1 \cap M_2) \subset Ass(M/M_1) \cup Ass(M/M_2) = \{p\}$.

So something minimal, you could collect them and get something smaller and still p_i-primary.

I'll talk about localization in a moment but I want to do examples. It's not hard to show the third one, but you still have to show something, that you can't have two p_i identical.

Let me describe a little bit what this gives. I'm going to be in characteristic zero and I'm going to do some geometry. What you should see here is that the minimal primes account for the geometric components. The primary have coefficients, and the nonprime ones are embedded stuff and are not unique.

Take $p \subset k[x_1, \ldots, x_n]$. Assume for ease that k is characteristic zero algebraically closed. We can look to p^n, which is not necessarily primary, but has a p-primary component denoted $p^{(n)}$. This is the kernel of the map going to the localization. p is a minimal prime over the annihilator of R/p^n. The annihilator is p^n. Then $p^{(n)}$ is the kernel of $R \to (R/p^n)_p$. This is unique; it's one of the minimal components; it's called the nth symbolic power of p.

What is the geometric meaning of this object? It's strange. Usually you have $p^n = p^{(n)} \cap \cdots$ It's something larger than p^n with more information.

Define for $n \geq 1$ that $p^{\langle n \rangle} = \{f \in R : f$ vanishes to order at least n at every point off $V(p)\}$. nonalgebraically, all polynomials vanishing together with all their partials of order $\leq n$ at all points on X. So it's $\cap_{x \in X} m_x^n$.

Please assume characterstic zero. Functions that vanish a lot, this is what I'm looking for. Now, Zariski and Nagata proved the following thing.

Theorem 2. *Zariski Nagata*
Over characteristic zero, algebraically closed fields, $p^{\langle n\rangle} = p^{(n)}$. for p a prime ideal.

What is an example where p^n is not equal to the symbolic power. Since we played with Macaulay let's come back to the same example. Take $R = \mathbb{C}[x_{ij}, 1 \leq i, j \leq 3]$. So I look to $M = (x_{ij})$ which I call the generic 3×3 matrix. I look to the ideal generated by the 2×2 minors of M. I will show that $\sqrt{I}$ is prime (actually I is). For those of you who know a bit, this is just defining the ... embedding of $P^2 \times P^2$.

I say that $p^2 \neq p^{(2)}$. The first of these is generated by sums of products of 2×2 minors. On the other hand $p^{(n)} = p^{\langle n\rangle}$. So the functions and their derivatives are in the ideal.

I claim that there is a polynomial in $p^{\langle 2\rangle}$ not in p^2. This is the determinant, in the symbolic square but not in the power. The partials of g, I need to show, depend on the minors. But $\partial det(x_{ij})/\partial x_{11}$ is the minor from eliminating x_{11}, and so on. You can show that $p^{(2)} \cap (x_{ij})^4 = p^2$.

There is no way to fill the gap without some geometry. I'm looking to the Segre embedding $\mathbb{P}^2\mathbb{C} \times \mathbb{P}^2\mathbb{C} \hookrightarrow \mathbb{P}^8$ where $y_0, y_1, y_2, z_0, z_1, z_2 \mapsto (y_0z_0, y_0z_1, \ldots, y_2z_2)$.

If you don't like coordinates this is $\mathbb{P}(V_1) \times \mathbb{P}(V_2) \hookrightarrow \mathbb{P}(V_1 \otimes V_2)$. What's true is that the image of this embedding is defined ideal-theoretically by the minors of a generic matrix. If I rename my variables $x_{ij} = y_{i-1}z_{j-1}$ then the 2×2 minors define the ideal of X. It's easy to show that this vanishes here, but the other direction is harder.

So $\{g = 0\}$ is a hypersurface in the symbolic square of the ideal. It's in $I(x)^{(2)}$ but not in $I(x)^2$. What is the geometric meaning? This has to do with resultants, amoebas, all these other things.

I can take the family of points on the line connecting two points on a variety. You have four degrees of freedom for each point in $\mathbb{P}^2\mathbb{C}$ and then a ninth along the line. So for any point in $\mathbb{P}^8$ there "should" be a line through that point described by two points in X. This is not true.

At a point of X I vanished all the 2×2 minors so by linear algebra the rank is one. The line connecting two matrices of rank one is a linear combination of these two, so has rank at most two. Then a point on the line is on the singular matrices.

A lot of information can be gotten from symbolic powers.