ALGEBRA III
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Last time we were talking about discrete valuation rings. When the Krull dimension was one, every nonzero ideal was a product of primary ideals with distinct associated primes.

Now if every primary is a power of a prime you got unique factorization. We looked at local rings with this property and there was a valuation. The trick is that these things are equivalent. Let me state the theorem, I won't prove all of it.

Theorem 1. *Let R be a local Noetherian domain of Krull dimension one. R/m is the residue field. The following are equivalent:*

(1) *R is a DVR.*
(2) *R is integrally closed in its field of fractions*
(3) *m is a principal ideal.*
(4) *$dim_k(m/m^2) = 1$. This is like the dimension of the tangent space.*
(5) *Every nonzero ideal is a power of m.*
(6) *There exists an $x \in R$ so that every nonzero ideal is of the type (x^k).*

What are examples? The things we did last time, like $\mathbb{Z}$ localized at a prime.

I won't prove all of these, but let me sketch some things. If you take nonzero $I \subset R$ then I contains a power of m. Try to show this. Every ideal in a Noetherian ring contains a power of its radical. This is not very hard once you write the generators.

Also note that $m^n \neq m^{n+1}$ for any n. This is Nakayama's lemma, since then $m^n = 0$. This is a contradiction for a number of reasons.

How do we show that a DVR in this situation is integrally closed? Any element in the fraction field that satisfies a monic equation with coefficients in R must be in R. So if $x \in K = Q(R)$ and $x^n + \cdots + a_n = 0$ then $x \in R$. So assume this; then $x \in R$ or $x^{-1} \in R$ and then x can be written as a sum $-(a_1 + a_2x^{-1} + \cdots)$ so is in R.

The only one which is still mysterious is that integrally closed implies principal. I want to show that for some $a \in m$, a generates m.

We know that $(a) \supset m^n$ for some least n. Take $b \in m^{n-1}$ but not in (a). So $a/b \in K$ but $b/a \notin R$. This is because $b \notin (a)$. So x^{-1} is not integral. Then $x^{-1}m \not\subseteq m$. If this happens, this is a faithful module over $R[x^{-1}]$, finitely generated as an R-module. So $x^{-1}m \subset R$ because this is $(b/a)m \subset m^n/a \subset R$. So $x^{-1}m = R$ so m is generated by x.

Okay, so that's it. Localizations of integrally closed rings are integrally closed. Powers of prime ideals remain powers of prime ideals through localization. So then we get:

Theorem 2. *Let R be a dimension one domain. The following are equivalent:*

(1) *R is integrally closed.*
(2) *Every nonzero ideal sia power of a prime.*
(3) *R_p is a DVR.*

You prove this with the last theorem and the fact that localization commutes with things. These are Dedekind rings. They may not be PID's, but their localizations are. You see these as algebraic closures of $\mathbb{Z}$ in an extension K of $\mathbb{Q}$.

Write $z^p = x^p + y^p$; in some Dedekind rings unique factorization proves Fermat for certain p.

The Dedekind ring, how do you see if it is a PID? An ideal in it is an R-module. A principal ideal look like (x); then $(x)(x^{-1}) = K$. When can I find a submodule of K which is an R-module, called N such that $MN = K$ for a given M. Every fractional ideal in a Dedekind ring has an inverse. So look to the group of fractionary ideals modulo the group of principal fractionary ideals; this is called the class group.

0.1. Nate's talk.

Definition 1. *A link is a collection of n oriented closed disjoint piecewise linear curves in S^3. If $n = 1$ then this is a knot.*

Two links are equivalent if there is an orientation preserving homeomorphism of S^3 taking one to the other.

One wants to find invariants of this object to show when these things are equivalent.

If L is a link, let X be the complement and we'll look at invariants of X. In some sense the best one is the fundamental group. You might wonder about the homology group, but the first homology of X if L has n components is $\mathbb{Z}^n$.

Let N be a thickening of L. You can write $S^3 = X \cup N$ and $X \cap N = T^2$. Now I can use a Mayer Vietoris sequence to show that if K is a knot then $H_1(K) = \mathbb{Z}$. The sequence I want is

$$H_2(S^3) \to H_1(X \cap N) \to H_1(X) \oplus H_1(N) \to H_1(S^3).$$

The first and last of these are zero; a torus has first homology $\mathbb{Z} \times \mathbb{Z}$, and a solid torus has first homology $\mathbb{Z}$. That shows it.

That ends up not being that great an invariant. I want to use it to construct something with more information.

I forgot to mention that a link diagram is a projection of a link into $\mathbb{R}^2$ I require that it be nice so that it is only $2 \to 1$ at finitely many points, is never $3 \to 1$, and no $2 \to 1$ points are vertices.

I can define the linking number $lk(K_1, K_2)$ as the number of times they cross one another, signed according to your favorite rule.

I want to use the homology group to construct something more interesting.

If I have a normal subgroup $H < \pi_1(X)$ then there is a cover $E_H \to X$ with fiber H. So $\pi_1(E_H) = \pi_1(X)/H = p_*(\pi_1(X)) = \Gamma(E_H, X)$.

So I want to construct the infinite cyclic cover of the complement of my link.

In this case I have the infinite cyclic cover. The idea is I have $\pi_1(X)$ with the commutator subgroup, so I have a cover X_∞ with $\pi_1(X_\infty) = \mathbb{Z}$. Furthermore the deck group is $\mathbb{Z}$. So $\mathbb{Z}$ acts on $\pi_1(X_\infty)$ so that it also acts on $H_1(X_\infty)$. It can also act naturally on it as an abelian group. Then it is a module over the group ring $\mathbb{Z}[t, t^{-1}]$.

Now I want to talk about the fitting ideals which are invariants of a presentation of a finitely generated module over a ring. If you go for a link, you will have as many variables as you have components of the link. So this becomes like studying modules over the Laurent polynomials.

Definition 2. *If F, G are free over R then a presentation of M is a sequence $F \to^\phi G \to M \to 0$.*

The most general way to write this is with the map $\tilde{\phi} : \wedge^j F \otimes \wedge^j G^* \to R$ where $(f_I, g_I^*) \mapsto g_I^*(\phi(F_I))$.

So then $Fitt_j(M)$ is the ideal which is the image of $\tilde{\phi}_{r-j}$.

Definition 3. *Given a knot K, the i Alexander ideal of K is the i Fitting ideal of $H_1(X_\infty)) \subset \mathbb{Z}[t, t^{-1}]$.*

A generator of the smallest principal ideal containing the fitting ideal is called the i Alexander polynomial.