MAT :Algebra III
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I'm not quite going to teach this class as a continuation of Algebra I and II. The main goal of the class would be to learn techniques in two areas: commutative algebra, including graded commutative, and representation theory. I'm not going to have time to touch everything, but depending on the interest of the students I may shift my focus somewhat. I want to focus on applications, so we'll discuss some algebraic geometry, combinatorics, etc. We'll also discuss computational and algorithmic techniques. We have a lab downstairs in S235 with roughly 30 linux boxes with a lot of mathematical software on them. I will mainly use two or three software programs, and sometimes we'll meet down there. I'll probably just use algorithms, rather than proving their correctness.

The webpage for the class will contain a lot of information. I won't follow any specific book; I'll probably combine several books. You also don't need any programming skills; everything will have syntax like mathematica or maple. The room down there is reserved during this time, and we will probably meet downstairs no more than 6 or 7 times. Some oof the software we'll be using includes:

- Macaulay 2
- Singular
- Maple/Mathematica (a little, these are not particularly efficient for what we'll be doing)
- GAP

There is a webpage with more information at http://www.math.sunysb.edu/ sorin/online-docs. We'll use Macaulay the most, the others here and there. Now, these are not used only for algebra, but also for analytic and topological questions phrased algebraically, for instance finding Betti numbers.

In terms of books and references, one book that you should try to read or browse in this class, which I will be following sometimes, is "Introduction to Commutative Algebra" by Atiyah

and MacDonald. It's hard to find, 90 dollars for an out of print paperback. The authors aren't algebraists.

Another book, which you should be able to get in the bookstore, is "Using Algebraic Geometry" by Cox, Little, and O'Shea. I'll use some things from this text, especially algorithms. I won't be focussing on the proofs. I have the old edition; the new edition will be coming out in a few weeks.

A third book, which is a reference book, is Eisenbud's "Commutative Algebra with a View toward Algebraic Geometry." This is a wonderful book, but it is over 600 pages, so we'll have to be selective. It has a broader horizon than say, "Using Algebraic Geometry," with which it overlaps. There is also a shorter book, Matsumura's "Commutative Rings." I like it a lot, more than Eisenbud's, but it is very dry. There are very few examples and little motivation.

To learn a little about algebraic geometry, you might want to look at the tiny blue London Mathematical Society publication by Hal Shenck, something like "Algebraic Geometry with a computational approach." Also Shafaraevich's algebraic geometry, just the first volume.

There's an undergraduate text "Ideals, Varieties, and Algorithms," also by Cox, Little, and O'Shea, an easier version of the other book by them.

I'll assign homework and problems in class every once in a while. Since this is how the department thinks about a 500 level course, the grade will be based on homework and a presentation. If you do all of the homework, you don't need to do a presentation; if you do an involved presentation, you don't need to do all of the homework. The problems could be theoretical, algorithmic, or in some other field. Now I want to know your background.

Where am I starting? How much algebra have you had? What year are you in?

- Tanveer Prince, in his 3rd year, with Algebra II
- Jun Ge, in his 2nd year, with Algebra II
- Mike Chance, in his 3rd year, with Algebra II
- Jaime Thind, in his 2nd year, with Algebra II
- Ning Hao, in his 2nd year, with Algebra II
- me, in my 2nd year, with Algebra II

Do you know what a noetherian ring is? Can you tell me a ring which is not? Hilbert basis theorem? Homological algebra?

1 Rings and Fields

We all know what rings are. Here a ring $(A, +, \cdot)$ is an abelian group under $+$ with a multaplicative monoid structure (i.e., associative multaplication with 1) which distributes

over addition. A field is a ring whose nonzero elements form an abelian group. If the group is not abelian we call it a skew field.

What are some examples of rings?

- $(\mathbb{Z}, +, \cdot), (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$
  These are a chain of subrings, which are subsets which are also rings, whose operations agree.
- $(\mathbb{Z}[i], +, \cdot)$ where the underlying set is $\{m + ni : m, n \in \mathbb{Z}\}$. This is a subring of $\mathbb{C}$ and contains $\mathbb{Z}$. This is not a field because, for instance, 2 cannot be inverted.
- $(\mathbb{Z}[\sqrt{d}], +, \cdot)$ where d is a squarefree integer.

There are often problems where you want to enlarge your ring. For instance, let ϵ be such that $\epsilon^2 + \epsilon + 1 = 0$. Then $\mathbb{Z}[\epsilon]$ is a ring. Suppose I want to prove that the fermat equation $x^3 + y^3 = z^3$ has no solutions. Well, if I was working on the squared case, I could write $x^2 + y^2 = (x + iy)(x - iy)$, factoring this over the Gaussian integers. If you are in a UFD, you are forced to have certain equalities. You can use this $\mathbb{Z}[\epsilon]$ to show that the fermat equation has no solutions other than the trivial ones.

Not every ring is a UFD. For example, in $\mathbb{Z}[\sqrt{-5}]$, you have $3^2 = 9 = (2 + \sqrt{-5})(2 - \sqrt{-5})$.

So these are rings of numbers. Here are some other examples.

- If k is a field, such as $\mathbb{R}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}$, with $+, \cdot$, then $k[x_1, \ldots, x_n] = \sum a_I \underline{x}^I$, where the sum is finite, $I = (i_1, \ldots, i_n)$, and $\underline{x}^I = x^{i_1} \cdots x^{i_n}$.
- A quotient $S/I = k[x_1, \ldots, x_n]/I$. This is a geometric object. Typically you want an algebraically closed field; let's choose $\mathbb{C}$.

Now we'll prove that every ideal is generated by finitely many elements, so that $I = (f_1, \ldots, f_m) = \{\sum_{i=1}^m g_i f_i | g_i \in S\}$. This corresponds to what is called an algebraic set $V(I) = \{z \in \mathbb{C}^n | f_1(z) = f_2(z) = \cdots = f_m(z) = 0\}$. On the other hand, beginning with a set $X \subset \mathbb{C}^n$, you can get an ideal $I(X) = (\{f \in S | f(X) = 0\}) \subset S$. This is an ideal of polynomials.

We can think of polynomial functions and their restriction to $V(I)$. Two functions coincide on $V(I)$ if and only if their difference vanishes there. So we think of the quotient as polynomials vanishing on $V(I)$.

We could look for instance at $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$, which is a circle. We want the field to be algebraically closed because $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ describes the empty set. We want the empty set to be described only by $0 = 1$.

Theorem 1 *Hilbert Basis Theorem*
Every ideal in $k[x_1, \ldots, x_n]$ is finitely generated,i.e., this ring is noetherian.

If I don't want to look at polynomials, I can look at the ring $\mathscr{C}(X, \mathbb{C}(\mathbb{R}) = \{f : X \to \mathbb{C}(\mathbb{R}) | f$ is continuous$\}$. Here X is a topological space. Then $(\mathscr{C}(X, \mathbb{C}), +, \cdot)$ is a ring. This is unpleasant only because it now has zero divisors. In a ring $(A, +, \cdot)$, an element $a \neq 0 \in A$ is a zero divisor if there exists $b \in A$ with $ab = 0$. A ring with no zero divisors is called an integral domain.

As long as you use the appropriate restrictions, the polynomial ring is an integral domain.

Now say you have $K \subset \mathbb{C}^n$ a compact subset. Let $A = \{f : K \to \mathbb{C} | f$ is the uniform limit on K of polynomials$\}$. This is again a ring, between the polynomial ring and the ring of continuous functions. How can we recapture the geometry of the polynomial ring? Pick a point $x_0 \in \mathbb{C}^n$. Take every function and evaluate at x_0, to get a ring homomorphism from A to $\mathbb{C}$. Then I can write an ideal as the intersection of kernels. Now is every ring morphism $A \to \mathbb{C}$ given by point evaluation? With a polynomial ring, the answer is basically yes, but not so with this A. Here you get $\bar{K} = \{z \in \mathbb{C}^n | \ |f(z)| \leq \sup_K |f|$ for all polynomials $f \in \mathbb{C}[z_1, \ldots, z_n]\}$.

Exercise 1 *Let $A = \{f : \{z \in \mathbb{C} || z| \leq 1\} \to \mathbb{C} | f$ is holomorphic for $|z| < 1$ and continuous on $|z| = 1\}$. Given $z_0 \in B_1, \phi_{z_0} : A \to \mathbb{C}$ is the evaluation map $f \to f(z_0)$.*
a. Show that every morphism $A \to \mathbb{C}$ is of this type.
b. Show that this is not true if we drop the requirement that f be continuous on the boundary $|z| = 1$.

The hint is to use fourier analysis on the boundary.

Exercise 2 *$A(\mathbb{C}) = (\{$ring of entire funtions on $\mathbb{C}\}, +, \cdot)$. This looks like it should be nice. Is $A(\mathbb{C})$ an integral domain? Is it noetherian? Even better, is it a PID? Here's a suggestion: use the Weierstrass multaplicative formula. The answer will be that this is not noetherian.*

Two more rings:

- The ring of linear differential operators on $\mathbb{R}^n$ with constant coeffiecients. Then $\mathbb{R}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}] \cong \mathbb{R}[t_1, \ldots, t_n]$.
- Say V is a vector space. Then $\wedge V = \oplus_{i=0}^{dim\ V} \wedge^i V, +, \wedge)$, which is not commutative, but graded commutative.
- $S(V) = \oplus_{i=0}^{\infty} Sym_i(V) \cong k[x_1, \ldots, x_n]$. Look at $k\langle e_1, \ldots, e_k \rangle$.

Where do these come from? Say X is a compact orientable manifold. Then we look at $(\oplus H^i(X, \mathbb{C}), \cup)$.
Now $\wedge V$ is very special; it is what is called an artinian ring.

Today I just gave a list of rings and properties. I'm going to start slowly, with some basic properties of ideals, of prime ideals and so on. I'll end with this.

Say I give you an ideal I of polynomials, and another polynomial g. Can you tell me whether $g \in I$?
Now, there's a theorem where if $V(I) = 0$ then $1 \in I$. So algorithmically, how do you write 1 as a combination of these three polynomials?