Introduction to Lie Groups and Lie Algebras
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Recall, for the classical groups we have the maps $\exp : gl(n) \to GL(n)$ and $\log : GL(n) \to gl(n)$. For G a Lie group we have $\mathfrak{g} = T_1 G$. We want some kind of analogue to the exponential map.

Last time we showed

Theorem 1 *Given $x \in \mathfrak{g}$ there exists a unique $\gamma_x(t) : \mathbb{R} \to G$ such that $\dot{\gamma}_x(0) = x$ and $\gamma_x(0) = 1$.*

Note that $\gamma_x(t) = \gamma_{x/\lambda}(t \cdot \lambda)$.

Definition 1 $\exp(tx) = \gamma_x(t)$. *This gives the map* $\exp : \mathfrak{g} \to G$.

Theorem 2
1. $\exp$ *is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of* 1 *in G.*
2. $\exp((t+s)x) = \exp(tx)\exp(sx)$
3. *If $f : G_1 \to G_2$ then $f_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfies the following:*

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "f_*"] \arrow[d, "\exp"'] & \mathfrak{g}_2 \arrow[d, "\exp"] \\ G_1 \arrow[r, "f"'] & G_2 \end{tikzcd}$$

4. *This diagram commutes:*

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "Ad\ g"] \arrow[d, "\exp"'] & \mathfrak{g}_2 \arrow[d, "\exp"] \\ G_1 \arrow[r, "h \to ghg^{-1}"'] & G_2 \end{tikzcd}$$

5. *Flow along the left invariant vector field defined by x over time t is $g = g(1) \to gF_t(1) = g\exp(tx)$.*
   Along the right invariant vector field it is $\exp(tx)g$.

The first of these is pretty easy. It is smooth because it is defined as the solution to a differential equation.
We certainly have $\exp(0) = 1$, and $\exp_* : T_0\mathfrak{g} \to T_1 G$, which can be identified as $\exp_* : \mathfrak{g} \to \mathfrak{g}$, is the identity because $\frac{d}{dt}exp(tx) = x$. The others I won't go through in detail. For the fifth one, you multiply on the right for a left-invariant vector field just because that's the action which commutes with the left invariant vector field. This is a little confusing at first. That isn't necessary at the moment, but will become quite useful.

Let's see what this looks like in various examples.

1. If $G \subset GL(n)$, then $\mathfrak{g} \subset gl(n)$, and the exponential map agrees with the map that was previously defined: $\exp(x) = \sum \frac{x^k}{k!}$. This will give a 1-parameter subgroup so by uniqueness this is the (only) solution.
2. If $G = \mathbb{R}$, then $\mathfrak{g}$ is $\mathbb{R}$ and the exponential is the identity. You can embed G in $GL(2)$ as $\{\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\}$ and then $\mathfrak{g} = \{\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}\}$. Then the exponential is just $\exp(X) = I + X$ which gives the result directly.
3. If $G - S^1 = \mathbb{R}/2\pi\mathbb{Z}$ or $z = e^{ix}$ then $\mathfrak{g}$ is still $\mathbb{R}$. The exponential map depends on how you write the elements: it is $x \to x \mod 2\pi$ or $x \to e^{ix}$.

Note that the exponential map is generally neither injective nor surjective. It is not surjective even for compact groups.

Corollary 1 1. *If G is connected, then G is generated by the elements* $\exp(x)$ *for x in some neighborhood of* 0 *in* $\mathfrak{g}$.

2. *If $a_1, \ldots, a_n \in \mathfrak{g}$ is a basis, then G is generated by* $\{\exp(t_1 a_1), \ldots, \exp(t_n a_n)\}$, *where t_n run over all sufficiently small real numbers.*
   Locally, $(t_1, \ldots, t_n) \to \exp(t_1 a_1) \cdots \exp(t_n a_n)$ *is a diffeomorphism, so that you can pick the word order.*
3. *Say G_1 is connected; then $Hom(G_1, G_2) \to Hom_{\mathbb{R}}(\mathfrak{g}_1, \mathfrak{g}_2)$ is an injection.*

Proofs:

1. The neighborhood is taken to a neighborhood of the identity, which we showed generates the connected component of the identity.
2. This follows easily from the theorem. I don't want to go into detail, you can check it for yourself. You combine it with the exponential map to get a neighborhood of the

identity in G, and then that generates the whole group. The hitch is that the product of the exponentials is not the same as the exponential of the sum. They're not the same maps but they have the same derivative at the origin, which is all that matters.

3. Suppose you have a morphism of Lie groups. Then this is determined by f_* uniquely by the third part of the previous theorem.

Look at $SO(3)$. Its Lie algebra $so(3) = \{a : a|a + a^t = 0\}$. I will choose the basis

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix};\ J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix};\ J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Leaving aside the reason I chose this basis, it's pretty obvious that these form a basis. Therefore in this sense they generate $SO(3)$. What is $\exp(tJ_x)$? This is rotation around the x-axis,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

You can do this explicitly, or you can take the derivative at $t = 0$ and see that you get J_x, which by uniqueness tells us that it is the exponential. Then the others are rotations about their respective axes. These generate the Lie algebra $\mathfrak{so}(3)$. So it suffices to give the value of any morphism from $SO(3)$ on these three generators. So if you want to describe the action of $SO(3)$ on a vector space, it is enough to describe how these vectors act, which is why these are used a lot by physicists. The general formulae are difficult, but the writing the images of the generators is not too hard.

I'll do this properly next time, but note that I never talked about any special structure on $\mathfrak{g}$. We don't really expect equality at the level of Hom, so how can we preserve the multiplication of G on $\mathfrak{g}$. For $x, y \in \mathfrak{g}$ we can write $\exp(x)\exp(y) = \exp(f(x, y))$. This map $f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ completely encodes multiplication locally. If you just compute the linear term then I claim you get $f(x, y) = x+y+\beta(x, y)+\ldots$, where the rest of the terms have degree three or higher. So this is the beginning of the Taylor series for f at 0 where β is bilinear and skew-symmetric.

It is common to write $\beta(x, y) = \frac{1}{2}[X, Y]$ to define the bracket.

Since $f(x, y) = \alpha_1(x)+\alpha_2(y)+Q_1(x)+Q_2(y)+\beta(x, y)+\ldots$ assuming you want $f(0, 0) = 0$. So how do you define these linear and quadratic terms? $f(x, 0) = x$ so $\alpha_1(x) = x$ and $Q_1(x) = 0$; similarly $\alpha_2(y) = y$ and $Q_2(y)$ is zero. So all that is needed to check is that the form is skew-symmetric. All you need to check is that $\beta(x, x) = 0$. Then $f(x, x) = 2x + \beta(x, x) + \ldots$ On the other hand, plugging in we get $\exp(x)\exp(x) = \exp(2x)$ so that $\beta(x, x) = 0$ and thus β is skew-symmetric. The punchline is that if I keep only the terms of degree 2 I get a skew symmetric form which makes $\mathfrak{g}$ an algebra.

Next time we'll talk more about the commutator, what it actually means, how to compute it.