Introduction to Lie Groups and Lie Algebras
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Recall that last time, we talked about actions of G on$M$, homogeneous spaces $M = G/H$, left, right, and adjoint actions of G on G, and left-invariant, right-invariant, and adoint-invariant vector fields.

Today we'll talk about classical groups and the exponential map. So first of all, what are the classical groups? This is used for the subgroups of the general linear group; for simplicity I'll talk over $\mathbb{R}$.

- $GL(n, \mathbb{R})$
- $SL(n, \mathbb{R})$
- $O(n, \mathbb{R})$
- $SO(n, \mathbb{R})$
- $U(n)$
- $SU(n)$
- $Sp(2n, \mathbb{R}) = \{A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} | \omega(Ax, Ay) = \omega(x, y)\}$ Here $\omega(x, y)$ is the unique (up to basis) skew-symmetric form $\sum_{i=1}^{n} x_i y_{i+n} - y_i x_{i+n}$. Or you can write it as (Jx, y) where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. There is a closely related group $Sp(n)$ which are the quaternion unitary transformations.

My main claim is that each of these classical groups is a Lie group, and we can find its dimension. We showed it for $GL(n)$ and $SU(2)$. For the general case you need tools. We'll use the exponential map. Now, $SO(n, \mathbb{R})$ is a subset defined by finitely many (n^2) equations. We want to show that they define a smooth manifold. You'd have to check the derivatives and their ranks and it would be a mess. We have an easier way.

Recall that $exp : gl(n) \to GL(n)$ takes x to $exp(x) = \sum_0^\infty \frac{x^k}{k!}$. Here $gl(n)$ is the set of all matrices. There's a locally defined inverse map which takes a neighborhood of the identity in $GL(n)$ into a neighborhood of 0 in $gl(n)$. This takes $1 + X$ to $\sum_1^\infty \frac{(-1)^{k+1} X^k}{k}$. Most of the properties are the same as for numbers; there are some differences. Note that this works for both the real and complex cases.

Theorem 1 *1. $log(\exp(x)) = x$; $\exp(log(X)) = X$ whenever they are defined.*

2. $\exp(x) = 1 + x + \dots$ This means $\exp(0) = 1$ and $d\ \exp = id$.

3. If $xy = yx$ then $\exp(x + y) = \exp(x)\exp(y)$. If $XY = YX$ then $log(XY) = log(X) + log(Y)$ in some neighborhood of the identity. This shows that $\exp$ is into $GL(n)$.

4. As a special case of the previous identity, if you consider $t \to \exp(tx)$ for some fixed x, then this agrees with the group operation. That is, $\exp((t+s)x = \exp(tx)\exp(sx)$. This transforms addition on the real line to multiplication of matrices, i.e., is a morphism of Lie groups. The images are very useful and are called one-parameter subgroups, which is bad terminology because they many not be submanifolds.

5. It agrees with many other operations, most importantly change of basis. That is, $\exp(AxA^{-1}) = A\exp(x)A^{-1}$.

I won't prove these. You usually do it by formal power series analysis. For the first, you just compute coefficients, and find the identity. The second is trivial. For the third, you just look at coefficients in the power series. The fourth follows from the third, and the fifth follows from $(AxA^{-1})^n = Ax^nA^{-1}$.

So that's all very nice and reasonable, but how are we going to use it? We identify some neighborhood of the identity in $GL(n)$ with some neighborhood of 0 in a vector space. Here is the key result about the classical groups.

Theorem 2 *For each classical group $G \subset GL(n)$, there exists a vector space $\mathfrak{g} \subset gl(n)$ such that*

$$(U \cap G) \underset{exp}{\overset{log}{\rightleftarrows}} (u \cap \mathfrak{g})$$

for some neighborhood U of 1 in $GL(n)$ and some neighborhood u of 0 in $gl(n)$, where each of the two (smooth) maps is a bijection.

So for instance if $G = GL(n)$ then $\mathfrak{g} = gl(n)$.

Corollary 1 *1. Each classical group is a Lie group.*

2. The tangent space at the identity $T_eG = \mathfrak{g}$.

Let's prove this corollary first because it's very easy. Well, near 1 you are associated with an open set in a vector space. So it is immediate that near 1, G is smooth.
If $g \in G$ then $g \cdot (U \cap G)$ is a neighborhood of g, which is therefore also smooth.

For the second part, how do we prove that? So you have the exponential map. Look at $\exp_*$ or $d\ \exp$, which will map from $T_0\mathfrak{g}$ to T_1G. Now, since $\mathfrak{g}$ is a vector space, it is its own tangent space, and since exp is invertible so is $\exp_*$. Since $\exp(x) = 1 + x + \ldots$, the derivative is the identity, so that the map is the identity.

How do we prove the theorem? It's done case by case.
For $GL(n)$ there's nothing to be done.
Suppose $X \in SL(n)$. Then $X = exp(x)$ for some x. How do we write the condition that $X \in SL(n)$? This means $\det\ X = 1$. So $\det\ \exp(x) = 1$ if and only if $e^{tr(x)} = 1$, i.e., if and only if $tr(x) = 0$ (put x in upper triangular form).

What about $O(n)$ and $SO(n)$? For O_n you need $XX^t = I$. Then these commute. Since the exponential map agrees with transposition, write these as $exp(x)exp(x^t) = I$. Then x and x^t should commute, so that $exp(x + x^t)$ is I, so that $x + x^t = 0$. Then the relationship is a linear condition $x^t = -x$.

What about for $SO(n)$? You have the same condition, but then you require that the determinant is 1. You get $tr(x) = 0$, which is unnecessary, because $x + x^t = 0$ implies that there are zeroes on the diagonal. So they share the same condition. This might seem confusing until you realize that $SO(n) = O(n)/\mathbb{Z}_2$.

You repeat a similar argument for the unitary group, except you replace transpose with adjoint.

For symplectic groups you have to do a couple more lines of work, which is what I ask you to do in the homework for $n = 4$.

This gives you more; it gives you dimension of the Lie groups because it gives you the tangent space at the identity.

G	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$O(n,\mathbb{R})$	$SO(n,\mathbb{R})$	$U(n)$	$SU(n)$	$Sp(2n,\mathbb{R})$
$\mathfrak{g}$	$gl(n,\mathbb{R})$	$tr\ x = 0$	$x + x^t = 0$	$x + x^t = 0$	$x + x^* = 0$	$x + x^* = 0,\ tr\ x = 0$	?
$\dim\ G$	n^2	$n^2 - 1$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	$n^2 - 1$	?
$\pi_0(G)$	$\mathbb{Z}_2$	$\{1\}$	$\mathbb{Z}_2$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\pi_1(G)$	$\{1\}$	$\{1\}$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}_2(n \geq 3)$	$\mathbb{Z}$	$\{1\}$	$\mathbb{Z}$

You should fill in the rest of the table.

Now we know that the universal cover has the structure of a Lie group. The universal cover of $SO(n,\mathbb{R})$ is called the spin group and is denoted $Spin(n)$. The easiest way to describe it is as the universal cover; since $\pi_1(SO(n,\mathbb{R})) = \mathbb{Z}_2$, this is a twofold cover.

The cover of the symplectic group is called the metaplectic group. This is a $\mathbb{Z}$-fold cover. This answers many of the questions we had about the classical groups.

Now we move to another question. Is there $exp : \mathfrak{g} \to G$ for a general Lie group (here $\mathfrak{g} = T_eG$)? You can't use a power series because we don't have multiplication. What can we say? The answer to the question is yes. To show this we have to go back to the properties of the exponential map, to see which could define this more general map.

The key idea is the one-parameter subgroup, which for the original exp was just $\exp(tx)$ as t ran over the real numbers.

Proposition 1 *Let $x \in \mathfrak{g}$. Then there exists a unique morphism of Lie groups $\gamma_x : \mathbb{R} \to \mathbb{G}$ with $\frac{d}{dt}\gamma_x|_{t=0} = x$.*

We bein with uniqueness. Now intuitively you'd like to write $\frac{d}{dt}\gamma_x|_t =$ "$\gamma_x(t) \cdot \frac{dx}{dt}(0)$" $=$ "$\gamma_x(t) \cdot x$". This is precisely $(L_{\gamma(t)})_* x$. It is equally obvious but slightly more cumbersome to write this as a right translation. This essentially gives you a differential equation for the map. Let v be a left-invariant vector field such that $v(1) = x$. Then γ is an integral curve for v.

For existence, let $F_t : G \to G$ be the flow for time t along the vector field v. Since my vector field is left invariant, my flow operator is also left invariant. Then $F_t(g_1 g_2) = g_1 F_t(g_2)$. Now let $\gamma(t) = F_t(1)$. Then $\gamma(t+s) = F_{t+s}(1) = F_s(F_t(1)) = F_s(\gamma(t) \cdot 1) = \gamma(t) F_s(1) = \gamma(t)\gamma(s)$, as desired.