Introduction to Lie Groups and Lie Algebras
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There is a new homework assignment, but I don't have the right number of copies.

So, let me remind you what we are doing. We are talking about representations of Lie groups and Lie algebras. We have two problems.

1. Decompose V into a direct sum of irreducibles. When is this possible?
2. Classify the irreducible representations (irreps)

Problem number two is much too hard for us at the moment. For almost every interesting case it requires tools we have not yet studied. So the only way to answer the second problem is by using Lie algebras. Today I would rather talk about the first problem. Assume you already have a classification of irreducibles. How do you decompose? This problem is possible to answer. Let's consider a case. Remember that all my representations are finite dimensional and complex unless otherwise stated.

Definition 1 *V is unitary if there is a positive definite inner product which is G-invariant $((\rho(g)v, \rho(g)w) = (v, w))$, or equivalently, $\rho(g) \in U(V)$.*

Example 1 *Let $V = F(S)$, complex valued functions on a finite set S. Let G be a finite group acting by permutations on S, then there is a pretty obvious G-invariant inner product. This is $(f_1, f_2) = \sum_s f_1(s)\bar{f}_2(s)$. This is a very special example.*

Why are we so interested in unitary representations? There is the following important result:

Theorem 1 *Each unitary representation is completely reducible, i.e., can be decomposed into a direct sum of irreducibles.*

The proof is pretty obvious, and goes by induction on the dimension. Either V is irreducible, and we're done, or V has a subrepresentation W. Then $V = W \oplus W^{\perp}$, and I claim that $W^{\perp}$ is a subrepresentation as well.

Say that $v \in W^{\perp}$; then $(v, w) = 0$ for all w in W. By the unitary condition $(gv, gw) = 0$ for all $w \in W, g \in G$. Then since g is invertible $gw : w \in W$ consists exactly of all the elements of w.

So what do we do now?

Theorem 2 *Any representation of a finite group is unitary.*

Start with some inner product $(,)$. We want $(gv, gw) = (v, w)$. This isn't likely, so we average. Define a new inner product $(,)^{\sim}$ as $\frac{1}{|G|}\sum_{g\in G}(gv, gw)$. I claim that this inner product is G-invariant. This is because $(gv, gw)^{\sim} = \frac{1}{|G|}\sum_{h\in G}(hgv, hgw)$, which is exactly $(v, w)^{\sim}$ because hg runs over G as h does.

There is one more thing to check, which is that this is positive definite. But this is positive definite as the sum of positive things.

Corollary 1 *Any representation of a finite group is completely reducible.*

Is there some way of getting the same result for some Lie group? For a Lie group we still have the notion of a unitary representation. But the existence of the inner product does not work in the same way because you can't take the sum. The obvious answer is that you replace the sum by the integral.

Theorem 3 *If G admits a Haar measure, i.e., a measure dg such that $\int_G dg = 1$ and dg is left, right invariant, then every representation of G is unitary.*

Define $(v, w)^{\sim}$ as $\int_G (gv, gw) dg$. We can discuss why this should be left and right invariant, instead of just right invariance. Let me leave it as a big question mark. Whether you can relax this condition, it seems you can but it can cause some problems. But anyway, given the hypotheses, we know that every representation is unitary, even if we can't calculate the integral.

Theorem 4 *Every compact Lie group admits a Haar measure. You can generalize this to compact topological groups with some additional conditions.*

For manifolds, there is a relation between top level differential forms and measures. If you have a manifold M^n and $\omega \in \Omega^n(M)$ a top level differential form which is nowhere vanishing, then $|\omega|$ is a measure.

So the problem of finding the measure can be solved if you can find a top level differential form.

Lemma 1 *To define a volume form on a group, you define it at the origin. At the origin this corresponds to something at the level of the Lie algebra. Any $w \in \wedge^n \mathfrak{g}^*$ is Ad G-invariant up to a sign.*

This fails for noncompact groups. So $\wedge^n \mathfrak{g}^*$ is one dimensional; then the action of G on it gives you a morphism $G \to \mathbb{R}^\times$. But the group is compact, so its image is compact; then the image is a compact subgroup of $R^\times$, which must be contained in $\{1, -1\}$. So it is $\{1\}$ or $\{\pm 1\}$. The group can only act on it by ± 1.

Here the action of G on $\wedge^n \mathfrak{g}^*$. The group acts on $\mathfrak{g}$ by the adjoint action, basically conjugation. This gives you an action on the dual. Then there's an action on tensor products of it. Once there's an action on tensor products, then the action preserves symmetric and skew-symmetric tensors. So it's actually rather simple. But basically all these structures are functorial so you'll always get induced morphisms.

So how do you extend this from the identity to the whole group? Choose your $\omega \in \wedge^n \mathfrak{g}^*$ and extend it to a left-invariant differential form on G. Then I claim that this thing will also be right invariant up to a sign. This is because the difference between left invariance and right invariance is just the adjoint action.

Thus $|\omega|$ is bi-invariant measure. But then you just renormalize by the measure of the group. So $dg = \frac{|\omega|}{\int_G |\omega|}$ is the Haar measure.

As an immediate corollary we have the following:

Corollary 2 *Every finite dimensional complex representation of a compact Lie group is completly reducible.*

So the compact groups are very nice. But it's only an existence result.

Example 2 *Consider S^1, which I think of as $\mathbb{R}/\mathbb{Z} = \{|z| = 1\}$. This is a compact group. What is the invariant measure? It's $d\phi$.*

So what are the irreducible representations of this group? That's a slightly more difficult question. This is a commutative group so by the Schur lemma every irreducible representation is one-dimensional. Start with an easier question. What are one dimensional representations of $\mathbb{R}$? Representations of it are the same as representations of the Lie algebra $\mathbb{R}$ with zero commutator. So what are representations of this? These are lines through the origin. $x \to \alpha x$ for $\alpha \in \mathbb{C}$. In the language of Lie groups it gives you the map $\exp(tx) \to \exp(t\alpha x)$. So this takes x to $e^{\alpha x}$.

So now to find the representations of S^1, which is not simply connected, watch which ones descend to the quotient. These are representations $\phi \to e^{\alpha\phi}$ such that integer values of ϕ yield the identity. So we want $e^{n\alpha} = 1$, which means $\alpha = 2\pi i k$ for $k \in \mathbb{Z}$.

So one dimensional representations of S^1 are parameterized by $k \in \mathbb{Z}$ and $\phi \to e^{2\pi i k\phi}$. It's a general principle that if you have a compact group then the irreducible representations are always discrete, i.e., there are only countably many.

Example 3 *Explicitly decomposing a representation of S^1 as a direct sum. Let $V = F(S^1)$, complex valued functions on S^1.*

This is an infinite-dimensional space so not everything will apply. S^1 will act on itself a rotation. How do you define a rotation invariant inner product? You can just integrate the product of two functions over S^1. That's the most obvious thing. $(f, g) = \int_{S^1} f\bar{g}d\phi$. The general theory suggests that it should be possible to decompose this into the direct sum of irreps. But what does that mean in the infinite dimensional case?

I claim that the answer is yes. I say that $V = \bigoplus_{k\in\mathbb{Z}} V_k$ where V_k is an irreducible one dimensional representation of S^1 on which ϕ acts by $e^{2\pi i k\phi}$. Here $V_k = \mathbb{C}e^{2\pi i k\phi}$. So this is a Fourier series. This explains why Fourier series show up; you want to see how things behave under the action of rotation. So you decompose it into things where the action of rotation is very simple.

So for example a rotation invariant differential operator must have these functions as eigenfunctions. And of course you know that Fourier series are used in many other places. The direct sum cannot really be used, because it's infinite. So we restrict ourselves to trigonometric polynomials with finite degree and finitely many terms. Or you can switch to the study of infinite dimensional representations and Hilbert spaces. This explains why study of Lie groups is referred to as noncommutative Fourier transforms. That's all I wanted to say today. Let's go to the office and I'll make copies of the homework. It will be due on October 12.