Introduction to Lie Groups and Lie Algebras
Lecture 8: September 28, 2004

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November 30, 2004

So, let's continue what we were doing. What we have finished is essentially the correspondence between Lie groups and Lie algebras.

Recall, there is a correspondence between connected, simply connected Lie group with Lie algebras (finite dimensional).

So what I'm going to do today is talk about representations, and probably for the next couple of weeks we'll be talking about representations of Lie groups and Lie algebras.

First recall the definitions. A representation of a Lie group is a map $\rho : G \to GL(V)$ for V a finite dimensional vector space.

A representation of a Lie algebra is a map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Here instead of preserving the product this preserves the commutator, $\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$.

What can we do with representations? First of all, there are some basic definitions.

1. A subrepresentation of G is a restriction $\rho \to GL(W)$ for a subspace W stable under the group action, or equivalently to $\mathfrak{gl}(W)$ at the level of Lie algebras.
2. There is a direct sum. I wouldn't bother you by defining how to get the map on the direct sum of two vector spaces if you know them on the individual ones. I hope you can do that yourself.
3. There is a trivial representation, if $V = \mathbb{C}$ then $\rho(g) = id$ or $\rho(x) = 0$. All of this should be over $\mathbb{C}$. There are good reasons, which I won't go into now.

   What else?
4. There is the dual representation on V^* It's a specialization of the slightly more general notion:
5. Tensor product. If you have two representations, you can form the tensor product of the vector spaces. The only problem is how you define the group action. You separate each component, so that $\rho(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$.

A more interesting question is, what happens at the level of Lie algebras? The action is defined as the derivative of a one-parameter subgroup corresponding to a given group element.

$$\rho(x)(v_1 \otimes v_2) = \frac{d}{dt}(\rho_1(\exp(tx))(v_1) \otimes \rho_2(\exp(tx))(v_2)).$$

You might expect $\rho(v_1 \times v_2)$ to be $\rho(x)(v_1) \otimes \rho(x)(v_2)$, but that's wrong.

It is an interesting exercise to check that ρ satisfies the Liebnitz equality. This is one, maybe two lines.

So how do we define the dual? We don't have too many choices. The action of G is defined so that $V \otimes V^* \to \mathbb{C}$ commutes with the action of G.

This defines how G should act on V^*. Why? If you have $v \otimes f$, then ρ takes them to $\rho(g)(v) \otimes \rho(g)(f) \to \langle \rho(g)(v), \rho(g)(f)) \rangle$.

So we have this identity $\langle \rho(g)v, \rho(g)f \rangle = \langle v, f \rangle$. This gives us $\langle w, \rho(g)f \rangle = \langle \rho(g^{-1})w, f \rangle$. Thus $\rho(g)$ on V^* is $\rho(g^{-1})^*$.

What do you think you should get for the action of a Lie algebra. If you have $V \otimes V^*$, and you act on the tensor product, then the following diagram should commute:

$$\begin{tikzcd}
V \otimes V^* \arrow[r] \arrow[d, "\rho"'] & \mathbb{C}\rho(x) \arrow[d] \\
V \otimes V^* \arrow[r] & \mathbb{C}.
\end{tikzcd}$$

This gives me $\langle v, \rho(x) f \rangle + \langle v, \rho \otimes f \rangle = 0$.This tells me how to compute in the Lie algebra by $\langle v, \rho(x) f \rangle = -\langle \rho(x) v, f \rangle$, or $\rho(x)$ on V^* is $-p(x)^*$.

So if B is a symmetric bilinear form $S^2 V^* \subset V^* \times V^*$.

This gives $(\rho(g)B)(u_x, u_y) = B(\rho(g^{-1})v_1, \rho(g^{-1})v_2)$, or $\rho \otimes B(v_1, v_2) = -B\rho(x)x_1, x_2 = -B(v_1, \rho(x)(v_2)$. In part, B is G invariant if and only if $V(\rho(g)v_1, \rho(g)v_2 = B(x, x_y)$, if and only if $\rho(x)B = 0$, if and only if $B(\rho(x)v_1, v_2) + G(x_1, \rho(x)(v_2)$.

Now $\rho(g)(A) = \rho(g)A\rho(g^{-1})$ in the particular case at work, with $End(V) = V \otimes V^*$. That is, $\rho(g)$ is just conjugation. Here $\rho(A)$ at the algebra level is $[\rho(x), A]$

Example 1 *1. $GL(n)$ acts on $\mathbb{C}^n$, hence on various tensors.*

2. *G acts on $\mathfrak{g}$ by Ad. The corresponding adjoint action is ad, where ad $x, y = xyx^{-1}$.*

   You also have an action on a group on a manifold. For example, $SO(n)$ acts on $C^\infty S^{n-1}$.

3. *G acts on $C^\infty(G)$. The examples are many, so I think we're going to see a lot of them.*

So what is main problem of representation theory?

1. Given a representation V, write it as the direct sum of "simple" ones. What exactly is a simple one? This is a problem that nmost of you know the answer to. You call it irreducible if it contains no subrepresentations.
2. Classify all irreducible representations
3. Classify all representations.

Not every V can be written as a direct sum of irreducibles (such are called completely reducible).

Example 2 *Let $G = \mathbb{Z}$. Let $n \to A_n$, with $A_n A_m = A_{n+m}$. Such a representation is completely determined by A_1, which can be any invertible map in $GL(V)$. This is because G is a cyclic group generated by 1. The problem of classifying all representations of G is equivalent to classifying all invertible operators. Then irreducibility would correspond to having no invariant subspaces. But then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has the subspace e_1, which gives a subrepresentation, but it cannot be split as a direct sum because then the matrix would be diagonalizable.*

The problem is that in many cases it's impossible to write the representation as a direct sum of irreducibles. Well, let's see what you can do.

Example 3 *Consider $GL(n)$ acting on $(C^n)^{\otimes 2}$ So can you split the space of rank two tensors into invariant subspaces under $GL(n)$. You can split $V^{\otimes 2} = S^2\ V \oplus \wedge^2 V$, each of which, I claim, is $GL(V)$-invariant. It is highly nonobvious whether these are irreducible, the answer in this case is yes. How did I come up with this decomposition? I won't tell you. Or rather, I will, but much later.*

We don't yet have much information, we don't know the answers of these questions. So let me explain why these are good questions.

One of them is actually the example I gave in the very first class. Say you have an operator on the sphere invariant under the action of $SO(3)$.

Definition 1 *Let V and W be representations of G. Then $\phi : V \to W$ is G-invariant (intertwining) if $\phi\rho(g) = \rho(g)\phi$. This is basically the same as equivariance.*

This is exactly what we had in the example in the first class, with $V = W = C^\infty(S^2), \phi = \Delta_{sph}, G = SO(3)$. In a lot of questions, it will appear as a group of symmetries of something. There is a large group whose action commutes with the action of the operator. Does it help us to understand this matrix, say, to compute its eigenvectors? The answer is yes. Let me start with this lemma:

Lemma 1 *Schur Lemma*

1. *Let V be an irreducible representation of G. Then the space of intertwining operators $Hom_G(V,V) = \mathbb{C}id$. So if an operator commutes with an irreducible representation then it is constant.*

2. *If V and W are irreducible representations which are not isomorphic then $Hom_G(V,W) = 0$.*

Let me give you a proof, it only takes two minutes. So, how you prove it? Let me, if $\phi : V \to V$ is intertwining, then the kernel of this operator and also the image are stable under the action of G. That is, they are subrepresentations. If $\phi v = 0$ then $\phi\rho(g)v = \rho(g)\phi v = 0$. So either the kernel is 0 and the image is the whole space or the kernel is V and ϕ is the zero map. This actually more or less gives you the proof of the second one, because you can consider $\phi : V \to W$. So that does part two. If they're not isomorphic then you only have the zero morphism.

So how do you know that every one of these isomorphisms in the $V \to V$ case is a multiple of the identity. Suppose ϕ is nonzero, and take λ an eigenvalue of ϕ. Consider $\phi - \lambda\ id$. This also commutes with the action of the group. Then again it must be zero or an isomorphism. Since it is singular it is not an isomorphism, so it is zero, which shows $\phi = \lambda\ id$.

Corollary 1 *We immediately get the following thing:*

1. *If $V = V_1 \oplus \ldots \oplus V_n$ and $V_i \ncong V_j$, then $Hom_G(V,V)$ is block diagonal:*

$$\begin{array}{c|c|c|c|}
 & V_1 & \cdots & V_n \\
\hline
V_1 & \lambda_1 & \cdots & 0 \\
\hline
\vdots & \vdots & \ddots & \vdots \\
\hline
V_n & 0 & \cdots & \lambda_n
\end{array}$$

   So this is what we should have done with the sphere, cut the maps up as the direct sum of irreducibles.

2. *If your vector space is the direct sum of irreducibles, not necessarily distinct, precisely $V = \bigoplus N_i V_i$, then $Hom_G(V,V) = \oplus Mat(N_i, \mathbb{C})$, via the embedding*

$$\begin{pmatrix} \lambda_{11}I & \cdots & \lambda_{1n}I \\ \vdots & \ddots & \vdots \\ \lambda_{n1}I & \cdots & \lambda nnI \end{pmatrix} \to \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda nn \end{pmatrix}$$

3. *If G is commutative then an irreducible representation is 1-dimensional. Suppose V is irreducible. Then for g in G we have $\rho(g) : V \to V$ is G-invariant implies $\rho(g) = \lambda id$. So any subspace is a subrepresentaion, being stable under the map. This forces one-dimensionality. For an abelian group, diagonalizing all group elements simultaneously is equivalent to diagonalizing all operators.*

4. *If G is a Lie group and $z \in Z(G)$, with a V-representation. Then V can be written as the direct sum $\oplus_\lambda V_\lambda$, where V_λ is the generalized eigenspace for z. Then this decomposition is stable under the action of G. If you have a central element, it gives you a way to split your vector space into smaller ones. It may not give you the full answer but it will give you something.*

Probably I should stop here. It's been pretty basic, and we still haven't progressed to our main goal, decomposing into a direct sum of irreducibles. I'll talk about that for compact groups next time, and we'll turn to Lie algebras to find out how.