Introduction to Lie Groups and Lie Algebras
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Last time we were discussing the correspondence between Lie groups and Lie algebras. We had

1. If G_1 is simply connected then $Hom(G_1, G_2) = Hom(\mathfrak{g}_1, \mathfrak{g}_2)$.
2. There is a bijection between immersed connected subgroups of G and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$. An immersed subgroup is just the image of a Lie group.

   The correspondences are obvious. It almost reduces all the questions about Lie groups to Lie algebras. There is still one problem left to discuss, and that is the following.
3. For every finite dimensional Lie algebra $\mathfrak{g}$, there is a Lie group G such that $\mathfrak{g} = Lie(G)$. Moreover, if we require G to be connected and simply connected then G is unique.

I'm not going to prove this theorem, because it relies on things we don't know, but the idea is as follows. It suffices to show that you can embed $\mathfrak{g}$ in $gl(n)$. Start with $\mathfrak{g} \to gl(\mathfrak{g})$. This takes x to $ad\ x$. You need to show that $ad\ [x, y] = [ad\ x, ad\ y] = ad\ xad\ y - ad\ yad\ x$. This is $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$, which is the Jacobi identity.

The only problem is that this is not an embedding. It has kernel $\{x|ad\ x = 0\} = \{x|[x, \mathfrak{g} = 0\}$. This is denoted $Z(\mathfrak{g})$ and called the center. The center of a Lie algebra corresponding to a Lie group is the Lie algebra of the center of the Lie group.

So this gives you a good starting point but it's not there. But you can write a Lie algebra as a semidirect product. One has trivial center and the other is in some sense like a commutative algebra; precisely it is solvable. Then you can use this method for the part with trivial center, and use other methods for the solvable half.

So you have to do things with simple and solvable algebras and semidirect products, and we don't really know how to do that. Oh, so to get it connected and simply connected just look at $\tilde{G}^0$, the universal cover of the component of the identity. For uniqueness, we can do better than that. We can describe all groups that have the same Lie algebra.

If G is connected and simply connected with $Lie(G) = \mathfrak{g}$, how can we describe all other connected Lie groups with the same Lie algebra? Use the first theorem to show that G maps onto any other connected Lie group with the same algebra. So $G' = G/\Gamma$, where Γ is a discrete normal subgroup. So G is central and typically you don't have many choices.

Example 1 *What are the connected Lie groups with Lie algebra $\mathfrak{su}(2)$?*

The answer is $SU(2)$ and $SO(3)$ since the center of $SU(2)$ is ± 1.

Corollary 1 *The category of connected simply connected Lie groups is equivalent to the category of Lie algebras.*

So basically all the questions we have about Lie groups can be answered about Lie algebras. I already said one example last time; there's an easy way to construct a map from $SU(2)$ to $SO(3)$ on the level of Lie algebras instead. This is related to the notion of representation.

Corollary 2 *Representations of a connected simply connected Lie group G (group actions on a vector space) are in bijection with representations of a Lie algebra $\mathfrak{g}$ (morphisms from $\mathfrak{g}$ to $gl(V)$ for a vector space V. Here we calculate $p([x,y]) = p(x)p(y) - p(y)p(x)$.*

Since G is connected and simply connected, $Hom(G, GL(V)) = Hom(\mathfrak{g}, gl(V))$. Now if G is a Lie group with Lie algebra $\mathfrak{g}$.

A question from the rabble: does every exact sequence of Lie algebras split? No, that is false.

Note: representations of a Lie group G which is not simply connected factor nicely through the universal cover as $G = \tilde{G}/n$, so that representations of G are exactly $rep(G) = \{p : \tilde{G} \to GL(V) | p(N) = id\} \subset rep(\tilde{G}) = rep(\mathfrak{g})$.

There is in particular, if two Lie groups have the same Lie algebra, even if they do not have the same representations, there is a relationship between them.

Everything I have said so far was over the reals. It can just about all be repeated for the complex case where Lie groups are complex manifolds and Lie algebras complex vector spaces.

Suppose that G is a real Lie group, and $\mathfrak{g}$ is a real Lie algebra. Consider its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. If you prefer, just take $\mathfrak{g} = \mathfrak{g} \oplus i\mathfrak{g}$. This is a complex Lie algebra. Now consider $G_{\mathbb{C}}$, the corresponding complex Lie group, i.e., the Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The picture looks like this: (these are not maps, just steps)

$$\begin{array}{ccc} G & \hookrightarrow & G_{\mathbb{C}} \\ \downarrow & & \uparrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}_{\mathbb{C}} \end{array}$$

This implies that a C^∞ Lie group is analytic. I don't want to spend much time on this. You don't have to do this because the exponential and the Campbell Hausdorff expression are analytic.

Example 2 *1.* $(GL(n,\mathbb{R}))_{\mathbb{C}} = GL(n,\mathbb{C})$

2. $(SO(n,\mathbb{R}))_{\mathbb{C}} = SO(n,\mathbb{C})$

3. what is $(SU(n))_{\mathbb{C}})_{\mathbb{C}}$*?*

So $\mathfrak{su}(n)$ are those elements a of $gl(n)$ with $tr(a) = 0$ and $a + \bar{a}^t = 0$. Then $\mathfrak{g}_{\mathbb{C}} = \{a + bi : a, b \in \mathfrak{su}(n)\} \subset Mat(n \times n, \mathbb{C})$. This is only an embedding because this is a special case. I leave it to you to check that this map is injective. Now if I forget about the trace zero condition then I get that the image consists of those matrices which can be written as the sum of a hermitian and a skew-hermitian matrix, that is, all of them.

Since we have the trace zero condition we're left instead with $sl(n,\mathbb{C})$, so that $(SU(n))_{\mathbb{C}} = SL(n,\mathbb{C}) = (SL(n,\mathbb{R})_{\mathbb{C}}$. So these real groups have different topologies (one is compact) and different Lie algebras. But after complexification they are identical.

Okay, let's see what this tells us about representations. If I'm talking about real representation I don't see why these two, say, should have the same representations. But if I have a real Lie group it still makes sense to talk about its complex representations. So what do I mean? I mean a Lie group G acting on a complex vector space analytically. This is the same as $Hom(G, GL(n,\mathbb{C}))$ considered as a real group. Then this is the same as $Hom_{\mathbb{R}}(\mathfrak{g}, gl(n,\mathbb{C}))$. This last is a real Lie algebra with dimension $2n^2$.

If I replace G by its complexification I get the same, $Hom_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}, gl(n,\mathbb{C}))$. So now I can go back from Lie algebras to Lie groups to get $Hom(G_{\mathbb{C}}, GL(n,\mathbb{C})$. Then all four of these are the same.

Example 3 *Complex representations of* $SU(2)$ *are the same as complex representations of* $\mathfrak{su}(2)$ *which are complex representations of* $\mathfrak{sl}(2)$*, which are complex representations of* $SL(2,\mathbb{C})$.

Why do we study representations? One reason would be because it helps us understand, say, functions on the sphere.

Remember that the homework is due on Tuesday.