Introduction to Lie Groups and Lie Algebras
September 14, 2004

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November 30, 2004

As far as the homework assignment goes, I am collecting homework assignment one and homework assignment two is available at my office or on the webpage later today. So let me start by making a small correction to what I said last time. There was a question about whether the exponential map from the Lie algebra to the Lie group was always surjective. The answer is no. For instance, $\begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of exp for $SL(2,\mathbb{R})$, but it is surjective for compact groups. Last time I said that for compact groups it might not be surjective, which is false.

So last time we talked about $\exp : \mathfrak{g} = T_1 G \to G$. We have $\exp(x)\exp(y) = \exp(f(x,y))$. Here $f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. This takes (x,y) to $x + y + 1/2[x,y] + \ldots$; this gives the bracket or commutator $[,] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$. This is skew-symmetric.

So from now on when I talk about a Lie algebra I mean a vector space with such a bracket. Let's talk about properties.

1. $\exp x \exp y \exp -x \exp -y = \exp [x,y] + \ldots$ where all other terms have degree at least 3.
   This is because $\exp x \exp y = \exp x + y + 1/2[x,y] + \ldots$
   and $\exp -x \exp -y = \exp -x - y + 1/2[x, y + \ldots$.
   Then the whole product is
   $$\exp x + y + 1/2[x,y] - x - y + 1/2[x,y] + 1/2[x + y + 1/2[x,y], -x - y + 1/2[x,y]$$
   $$= \exp [x,y] + 1/2[x + y, -x - y] + \ldots = \exp [x,y] + \ldots$$
   So this measures noncommutativity to some degree.

2. if $f : G_1 \to G_2$, then the induced map f_* preserves the commutator in the sense that $f_*[x,y] = [f_*(x), f_*(y)]$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g}_1 \times \mathfrak{g}_1 \arrow[r, "f_* \times f_*"] \arrow[d, "{[,]}"'] & \mathfrak{g}_2 \times \mathfrak{g}_2 \arrow[d, "{[,]}"] \\ \mathfrak{g}_1 \arrow[r, "f_*"'] & \mathfrak{g}_2 \end{tikzcd}$$

3. Given an adjoint action from an element of the Lie group, $Ad\ g : \mathfrak{g} \to \mathfrak{g}$, we have $Ad\ g[x, y] = [Ad\ g(x), Ad\ g(y)]$, or the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "Ad\ g \times Ad\ g"] \arrow[d, "{[,]}"'] & \mathfrak{g} \times \mathfrak{g} \arrow[d, "{[,]}"] \\ \mathfrak{g} \arrow[r, "Ad\ g"'] & \mathfrak{g} \end{tikzcd}$$

We want to show that we have a bijection between morphisms of the Lie group and of the Lie algebra, but we probably won't get there today.

There's one more important identity. We don't really have associativity, but this gives an identity satisfied by the commutator. This is the Jacobi identity.

Theorem 1 *Define $ad\ x : \mathfrak{g} \to \mathfrak{g}$, for $x \in \mathfrak{g}$, as $ad\ x.y = [x, y]$. Then $ad\ x$ is a derivation, so that $ad\ x.[y, z] = [ad\ x.y, z] + [y, ad\ x.z]$. So this is nothing but the Liebnitz rule.*

There are many ways of writing this identity, for example $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$. Using the skew symmetry we can rewrite this in a variety of ways. Probably the shortest is that the sum over every cylic permutation of $[x_1, [x_2, x_3]]$ is zero.

So how do we prove this? We want to find a relation between Ad (the adjoint action of an element of the Lie group) and ad (the adjoint action of an element of the Lie algebra). So look at $Ad\ \exp(tx)(y)$. I claim that this is $y + t[x, y] + \dots$

We check this by finding a curve with the correct tangent and applying a conjugation. The easiest curve with y as a tangent vector is $\exp sy$. Then the Ad action is $\frac{d}{ds}|_{s=0} e^{tx} e^{sy} e^{-tx}$. So by the first property in my list, I have

$$e^{tx} e^{sy} e^{-tx} = e^{sy} + ts[x, y] + \dots$$

Then the derivative at $s = 0$ is $y + t[x, y] + \dots$ Therefore ad is just the derivative of Ad.

Given an action $p : G \to End(V) = GL(V)$ this defines a map $p_* : \mathfrak{g} \to gl(V)$. This formula tells me that for $p = Ad$, we have $p_* = ad$.

Now, to see that we have the Jacobi identity, we use the fact

$$Ad\ \exp tx[y, z] = [Ad\ \exp tx(y), Ad\ \exp tx(z)].$$

This already implies that the linear term should be a derivation.

Take the derivative of this identity $\frac{d}{dt}|_{t=0}$. This gives $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ directly, as

$$\frac{d}{dt}|_{t=0}[Ad\ \exp tx(y), Ad\ \exp tx(z)]$$
$$= [\frac{d}{dt}|_{t=0}Ad\ \exp tx(y), Ad\ \exp tx(z)] + [Ad\ \exp tx(y), \frac{d}{dt}|_{t=0}Ad\ \exp tx(z)].$$

Definition 1 *A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity.*

So now we can ask about the relation between abstract Lie groups and abstract Lie algebras. For instance, there are Lie algebras which cannot be found as the tangent space at the identity of any Lie group.

Let's look at examples.

1. if $G \subset GL(n)$ then $\mathfrak{g} \subset gl(n)$. Then $[x,y] = xy - yx$.
   As far as the proof goes, you can start by writing $\exp x \exp y \exp -x \exp -y$. If you kill the terms with x^2 and y^2 you get $(1+x)(1+y)(1-x)(1-y)$, so that after a little high school algebra, you get that this is $1 + xy - yx + \dots$

2. Let's talk about $G = Diff(M)$. This is an infinite dimensional group. For the moment let's stay vague, and leave the rigor for later. So what is the analogue of the Lie algebra for this? Think of it as the tangent space at the identity. In this case what do we get? We look at all 1-parameter families of diffeomorphisms, and we take the derivative with respect to t.

   So we start with $\varphi_t : M \to M$ and then $\frac{d}{dt}\varphi_t(m) = T_m M$. So then $fracddt\varphi_t$ is a vector field on M. Let's take all of this as motivation and define the Lie algebra of this group to be the space of all vector fields. In this case, if you have $\xi \in Vect(M)$, what is $\exp(t\xi)$? It is just the flow of that vector field. It is supposed to be a one-parameter family of diffeomorphisms whose derivative is just my vector field. So this is the solution to a differential equation, which is just the flow along the vector field. This may not be defined globally, so there might be problems, but let's just ignore that for a little.

   For the moment let me leave it vague.
   So what is the commutator $[\xi, \eta]$?
   It is $F_t(\xi)F_s(\eta)F_t(-\xi)F_s(-\eta)$. You don't end up at the same point, but you aren't very far off. In some chart this will be $id + ts[\xi,\eta] + \dots$, by definition. This is used for a lot of differential geometry. You can describe it as the Lie derivative of one vector field along another. I'd rather not get into that because it's really differential geometry.

   Every vector field gives you a way of differentiating a function; then you can define the commutator of two vector fields this way: If δ is the differential operator, then $\delta_{[\xi,\eta]} = -(\delta_\xi\delta_\eta - \delta_\eta\delta_\xi)$. Many texts will leave out the negative sign, but it is necessary for some of our identities to work without negative sign. Note that this is not obviously a vector field.

Now let me make a precise statement. Suppose that you have a finite dimensional Lie group G acting on M. That is, you have a map $\rho : G \to Diff(M)$. Then firstly, $\rho_* : \mathfrak{g} \to Vect(M)$ is well-defined, and secondly, this agrees with the commutator in the following sense: $\rho_*[x, y] = [\rho_*(x), \rho_*(y)]$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \times \mathfrak{g} \arrow[r, "\rho_* \times \rho_*"] \arrow[d, "{[,]}"'] & Vect(M) \times Vect(M) \arrow[d, "{[,]}"] \\
\mathfrak{g} \arrow[r, "\rho_*"'] & Vect(M)
\end{tikzcd}$$

This is straightforward. There is a very special example, or I should say, special case. G acts on G by left multiplication. Then, according to this general statement, every element of the lie algebra $\mathfrak{g}$ should give you a vector field on G. So what is the vector field from $x \in \mathfrak{g}$? You find a one-parameter family of group which has x as its tangent vector. It takes x to the right invariant vector field corresponding to x. This is because left translation is right invariant.

Consider a one-parameter family of elements with x as derivative. The easiest is $\exp tx$. So we should look at $\frac{d}{dt}|_{t=0} \exp txg = \xi(g)$. If it made sense I'd just say this was xg, but in a general (nonmatrix) group you can't multiply elements of the Lie group and Lie algebra. So I write $(Rg^{-1})_* x$ which I will write xg. So this is how you find the vector field corresponding to an element of the Lie algebra.

Thus the commutator in $\mathfrak{g}$ agrees with the commutator of right-invariant vector fields on G.

So let me finish it with the following thing. I already said that if my group is commutative, then the commutator in the Lie algebra is zero. If the exponentials of two particular elements commute, then they commute in the Lie algebra, even. The question is whether we can reverse this. If x and y commute in the Lie algebra, do $\exp x$ and $\exp y$ commute in the Lie group? For matrix groups we know the answer is yes, but could there be higher degree terms that don't vanish in an abstract Lie group? The answer is no.

Theorem 2 *If* $[x, y] = 0$, *then* $\exp x \exp y = \exp x + y = \exp y \exp x$

Theorem 3 *This is called the Campbell Hausdorff Dynkin Theorem.*
So $\exp x \exp y = \exp f(x, y)$. *Here* $f(x, y) = x + y + 1/2[x, y] + \sum_{n>2}$(*some repeated commutators of* x, y *with* n *terms with rational coefficients*).

This is a universal formula which does not depend on the group, but it would not help you at all. So It's clear that the second theorem implies the first, because any repeated commutator will be a nested commutator with 0 and thus be 0. This is a series with a positive radius of convergence so that it contains a neighborhood of the origin.

Let me give you the idea of the proof. Write $\exp tx \exp y = \exp z(t)$ for $z : \mathbb{R} \to G$. If $t = 0$ then we know $z(0) = y$. So we can write the differential equation easily. We have the initial condition

$$\frac{d}{dt} \exp z(t) = x \exp tx \exp y = x \exp z(t).$$

For a nonmatrix group, technically again this is the right invariant thing, but I won't get into that notation.
Next $\frac{d}{dt} z(t) = (\exp_*)^{-1} \cdot x \exp z(t) = \sum_0^\infty B_k(ad\ z(t))^k \cdot x$.

So we can solve this with the standard tools, and it takes some effort, but it gives us an explicit formula. It should be obvious without going through all of that, every term will vanish except for the constant term in a commutative group.

I don't want to go into any more detail than that. I just want to show you that every term can be gotten by differential equations.

The first theorem has easier proofs. For instance, if you know that two vector fields commute, then their flows commute since you can look at them locally as translation in x and y in some coordinate system. But let me stop there since I don't know how much you know.

So first of all I want to see homework number one, and second I left homework number two in my office, so let's all go and pick it up from there.