Introduction to Lie Groups and Lie Algebras October 7, 2004

Gabriel C. Drummond-Cole

November 30, 2004

Since the time is already right for starting, let me talk about the homework. Most of you did pretty well, but some of you messed up between $SO(3)$ and $SL(2)$. The thing that most people couldn't do was the last problem, which is not surprising, because it's a messy problem if you don't do it right.

Remember, $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$.

[Stas: What does it mean, "action by vector fields?"]

Every element gives you a vector field, and the commutator respects that of the space of vector fields. Whether this is an action is debatable.

The first problem was to get things in terms of x, y, z. You do calculations, just be careful to expand with the Liebnitz rule. But that you probably all know so I don't think there's a problem.

The next problem is show that to compute this on the sphere you only need to calculate its values on the sphere. But spherical coordinates are a really bad mess. But you notice that $z\partial_y - y\partial_z$ is tangent to a rotation. If you apply this to e^{tx} you get only things on the sphere. So if you have $\zeta \in TS^2$ then $(\partial_\zeta f)|_{S^2}$ only depends on $f|_{S^2}$. So then the linear combinations of them have the same property.

As far as the third and fourth part go, that is, in writing the usual Laplace operator as a sum of $1/r^2$ times this and a radial operator, and to show this is rotation invariant, the easiest way is to do it in reverse order. The only thing you need is that $r\partial_r = x\partial_x + y\partial_y + z\partial_z$. It's not overly pleasant but it's not that bad.

It is known and easy to show that the usual Laplace operator is rotation invariant, and then the radial component is rotation invariant. Then the linear combination of these is rotation invariant.

There's another way to do this but you don't know it yet.

[Zeng: ???!]

You write $\mathbb{R}^3\backslash\{0\}$ as $\mathbb{R}_+ \times S^2$ and you find that $r^2 \times \Delta$ splits into a sum of second order differentials only in one and only in the other direction.

In this case it turns out that this splits into two terms.

[Zeng: !?!???!?!!]

!?@??@????

[Zeng: !?!?!????!!!!!??!?!!??!]

No, I didn't say you could do that. The key point here is the action of the group, rather than just having a direct product.

So the question is whether you can actually show that this thing is rotation invariant without doing all of this. The answer is, yes you can.

One way would be with explicit calculations. What is the adjoint action of g on its own lie algebra. It will be something like $gJ_xg^{-1} = \sum g_{i1}J_i$, and similarly for J_y and J_z. If you take this whole expression you get something like $\sum_j(\sum_i g_{ij}J_i)^2$. These J_i don't commute, but after some computation you get 27 terms with no partial derivatives. All you need is the commutator relation. This is not a pleasant way but it's manageable. It's better than computing partial derivatives.

So let's move to how to do it even better, which is the main topic here.

So far we've been talking about representations and characters. But nothing is practical because you end up with a nontrivial combinatorial problem. All this gave us no idea how to classify representations. How do we decompose these.

From now on let's forget Lie groups and instead talk about Lie algebras.

Definition 1 *A Lie algebra is a vector space with* $[,]$ *which is skew symmetric and satisfies the Jacobi identity.*

We know that every finite dimensional Lie group gives a Lie algebra, and vice versa, but we don't care about that right now.

so we can do the standard things we do. We can define subalgebras, ideals, and so on. An ideal $I \subset \mathfrak{g}$ is a set such that $[\mathfrak{g}, I] \subset I$. If you have an ideal you can define a quotient algebra $\mathfrak{g}/I$. I don't feel like I really need to prove it.

We also talked about representations of a Lie algebra.

Definition 2 *A representation is* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *which agrees with the commutator in* $\mathfrak{gl}$.

As a matter of fact you can do this to construct a Lie algebra from an associative algebra. More generically you can define a map from $\mathfrak{g} \to A$ for A an associative algebra, when you define a commutator in A as $[a, b] = ab - ba$.

So it turns out that there is a universal associative algebra associated with it, called the universal enveloping algebra.

Definition 3 *The universal enveloping algebra* $U_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *is the associative algebra with* 1 *generated by elements* $a_x, x \in \mathfrak{g}$, *and the relations* $a_{[x,y]} = a_x a_y - a_y a_x$.

So you take your Lie algebra and construct out of it an associative algebra. Why do we need such an object? Because if we have a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, then we can have operators $\rho(x_1)\rho(x_2)\cdots\rho(x_n)$. There is never a J_x^2 in $SO(3)$, but you can find a meaning for it over a vector space. So it makes sense to consider the universal object of this kind. Impose the obvious relations, and what are the things you get? What I said can be made precise as follows:

Theorem 1 *Given a representation* ρ *of* $\mathfrak{g}$, *there is a unique map of associative algebras* $U_{\mathfrak{g}} \to End(V)$. *Conversely, given a map of associative algebras* $U_{\mathfrak{g}} \to End(V)$ *it defines a representation of* $\mathfrak{g}$ *in* V.

Then V becomes a module over $U_{\mathfrak{g}}$. It's a trivial thing, a tautology.

Elements of $U_{\mathfrak{g}}$ are sums of products. You know how elements of $\mathfrak{g}$ should act and you know that this should be a map of algebras. So $a_{x_1}\cdots a_{x_n}$ maps to $\rho(a_{x_1})\cdots\rho(a_{x_n})$. I leave it to you to see how it goes the other way around.

Also, we'll frequently write $x_1 \cdots x_n \in U_{\mathfrak{g}}$ instead of $a_{x_1}\cdots a_{x_n}$. For example, $J_x^2 + J_y^2 + J_z^2 \in U_{\mathfrak{so}(3)}$ or $e^2 + 2hf \in U_{sl(2,\mathbb{C})}$. If you have a representation of $\mathfrak{sl}(2,\mathbb{C})$ then this multiplication makes sense; this does not occur in $\mathfrak{sl}(2,\mathbb{C})$.

In any finite dimensional representation e is nilpotent, but not in U, which has no universal dimension.

So $U_{\mathfrak{sl}(2,\mathbb{C})}$ is $\langle e, f, h\rangle / \{he - eh = 2e, ef - fe = h, hf - fh = -2f\}$. This actually does not imply $e^2 = 0$, for example if you construct three-dimensional representation. For example, if you do the symmetric powers in the homework, but that we'll do later.

The easiest intepretation of Δ_{sph} is that it is in the universal enveloping algebra so it is an element of any representation.

Oh, one more thing, if G acts by diffeomorphisms on M, then we know that G defines vector fields. What is the natural interpretation of $U_{\mathfrak{g}}$? They are formal products of elements of $\mathfrak{g}$ These are differential operators, i.e., the linear combinations of images of formal products of first order differential operators.

Why is this useful? Central elements in the Lie group act by a constant in any representation. But there aren't many of them. Instead of talking about central elements of Lie group we can talk about central elements of the Lie algebra, but that is no better. But how about central elements of $U_{\mathfrak{g}}$?

Definition 4 *$Z(U_{\mathfrak{g}})$ is the center of $U_{\mathfrak{g}}$.*

Other definitions: on $U_{\mathfrak{g}}$ we have the adjoint action of G by the only reasonable extension. Then we can describe the same thing for the adjoint action of $\mathfrak{g}$, i.e.,

$$ad\ g.(x_1 \cdots x_n) = (ad\ gx_1)x_2 \cdots x_n + \cdots + x_1 \cdots (ad\ gx_n).$$

This is

$$(gx_1 - x_1 g)x_2 \cdots x_n + \cdots + x_1 \cdots (gx_n - x_n g).$$

So there's a lot of cancellation and you get just the first and last terms, $gx_1 \cdots x_n - x_1 \cdots x_n g$. So $ad\ x\ a = xa - ax$ for all $a \in U_{\mathfrak{g}}$.

Lemma 1 *The center of $U_{\mathfrak{g}}$ is $(U_{\mathfrak{g}})^G = (U_{\mathfrak{g}})^{\mathfrak{g}}$.*

The second and third are the same in any representation. constants are those functions whose derivatives are 0. It is clear that a central element must vanish under $ad\ x$, and conversely if you vanish under $ad\ x$ then you commute with x so invariance under ad means commutativity with every algebra element.

Exercise 1 $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2 \in Z(U_{\mathfrak{so}(3)})$.

So

$$ad\ J_x \Delta_{sph} = [J_x, J_y]J_y + J_y[J_x, J_y] + [J_x, J_z]J_z + J_z[J_x, J_z] = J_z J_y + J_y J_z - J_y J_z - J_z J_y = 0.$$

Corollary 1 *Whenever you have a Lie algebra representation of $\mathfrak{so}(3)$, this operator commutes with the action of $SO(3, \mathbb{R})$ (if this is an action of $SO(3, \mathbb{R})$ So in any irreducible representation, it acts by a constant.*

So you see that $SO(3)$ and $\mathfrak{so}(3)$ have no central elements, but $U_{\mathfrak{so}(3)}$ does. So diagonalize this representation and you'll get subrepresentations.

We have about ten minutes so let me finish by the following thing.

You can construct central elements in the universal enveloping algebra for many lie algebras, for instance for $\mathfrak{sl}(2, \mathbb{C})$ and for $\mathfrak{gl}(n)$.

For every simple Lie algebra there are sufficiently many central elements in the universal enveloping algebra to classify irreducible representations. This is too long a story for right now.

Let me say a little about the size of the universal enveloping algebra. How large is $U_{\mathfrak{g}}$? Let me first start

1. If $\mathfrak{g}$ is commutative, i.e., $[,] = 0$ then $U_\mathfrak{g}$ is $\langle x \in \mathfrak{g}\rangle / xy - yx = 0 = S\mathfrak{g}$, or polynomial functions on $\mathfrak{g}^*$. This is a polynomial algebra, which immediately answers some of the questions. This is infinite dimensional. What is the dimension of the degree n component? You know how to find it.

2. What is the general case? In general, it has the same size in the commutative case, which doesn't make much sense since they're infinite dimensional. One way of making it precise, which involves choosing a basis, involves monomials.

We specify an order on x_i and then get our monomials in the right order, by, say, replacing x_1x_2 with $x_2x_1 + [x_1, x_2]$. So we don't need both of these in a basis if we're building up by degree.

Theorem 2 *Poincaré=Birkhoff-Witt (PBW)*
Choose a basis $x_1, \ldots, x_n$ in $\mathfrak{g}$; then the mnonomials $x_{i_1} \cdots x_{i_k}$ for $i_1 \leq \cdots \leq i_k$ form a basis in $U_\mathfrak{g}$. These span $U_\mathfrak{g}$.

So any product can be written as a sum of these. It's easy to show that these span; it's very hard to show that they are linearly independent. You really need the Jacobi identity. If I were to do this and I was stupid enough to choose an algebra without the Jacobi identity, this quotient would be 0.

The infinite dimensionality doesn't cause any problems.