Introduction to Lie Groups and Lie Algebras
October 5, 2004

Gabriel C. Drummond-Cole

November 30, 2004

Today we're talking about characters and the Peter-Weyl Theorem.

Recall that every finite dimensional complex representation of a compact Lie group is unitary and thus completely reducible.

Lemma 1 *1. Say V and W are irreducible with $V \nsim W$, and $f : V \to W$. Then $\int_G gfg^{-1}dg = 0$.*

2. If $f : V \to V$ then $\int gfg^{-1}dg = \frac{tr(f)}{dim\ V} id$.

Proofs. Let $\tilde{f}$ be the initial integral. This commutes with action of g, i.e., $h\tilde{f}h^{-1} = \int_G (hg)f(hg)^{-1}dg$.

Then the Schur lemma gives that $\tilde{f} = 0$ for $W \neq V$ and $\tilde{f} = \lambda id$ for $W = V$. Applying this in the particular case we get the second part of the lemma.

Let e_i be an orthonormal basis for a G-invariant inner product $(,)$. Take f to be E_{ij}, i.e., the operator which maps e_i to e_j, and everything else to 0. Then $gfg^{-1}e_k = g_{jk}^{-1}gfe_j = g_{jk}^{-1}ge_i = \sum g_{li}g_{jk}^{-1}e_l$. Then when you plug in you get

$$(\int gfg^{-1}dg)e_k = (\sum_l \int g_{li}g_{jk}^{-1}dg)e_k.$$

But on the other side this is easy to find because it is a multiple of the identity. So this is $\frac{\delta_{ij}}{dim\ V}e_k$. So what's the point? The point is that if I compare formulas I get the following important relation:

$$\int g_{li}g_{jk}^{-1}dg = \frac{\delta_{ij}delta_{kl}}{dim\ V}.$$

So because of inner product considerations we know $g^{-1} = \bar{g}^t$ so we can write this

$$\int g_{li}\bar{g}_{kj}dg = \frac{\delta_{ij}\delta_{kl}}{dim\ V}.$$

This gives you something a lot like orthonormality. What we've got is the following result.

Theorem 1 *1. Let V be irreducible and finite dimensional with basis $\{e_i\}$. Then the $(dimV)^2$ functions g_{ij}^V are orthogonal and*

$$|g_{ij}^V|^2 = \frac{1}{dim\ V}.$$

(Here V is upper index to show that g is acting by matrix on V. They're orthogonal and almost orthonormal.

2. If V, W are nonisomorphic and irreducible then g_{ij}^V and g_{kl}^W are orthogonal. So each irreducible representation gives you $(dim\ V)^2$ which are all orthogonal.

In particular, they are all linearly independent.

So why is this of any use? Matrix elements alone are not nice objects because they depend on choice of basis. But there is one combination of them which doesn't depend on choice of basis.

Definition 1 *Let V be a representation. A character of V is $\chi_V \in C^\infty(G)$ defined by $\chi_V(g) = tr_V g$. This is the trace of the representation, not of the original g if it was a matrix.*

Example 1 *1. $V = \mathbb{C}$, the trivial representation. Then $\chi_V = 1$.*

2. $\chi_{V\oplus W} = \chi_V + \chi_W$

3. $\chi_{V\otimes W} = \chi_V \chi_W$ That's not so good, since $V \otimes W$ is a very large space.

4. $\chi_V(ghg^{-1}) = \chi_V(h)$.

The third one one you can prove basically by diagonalizing your group element. Then you find eigenvalues of the product as products of the eigenvalues.

So why is it of any use? It's canonically defined. If I were to choose a basis I could define it in terms of matrix elements $\chi_V(g) = \sum g_{ii}^V$.

So from here to find my characters I can use a one-line manipulation.

Theorem 2 *1. If V, W are nonisomorphic irreps, then $(\chi_V, \chi_W) = 0$. Here the inner product is $\int \chi_V(g)\overline{\chi_W(g)}dg$. This is because they are the linear combinations of matrix elements, which are individually orthogonal.*

2. $|\chi_V|^2 = \sum |g_{ii}^V|^2 = 1$.

So this is a tool in computing irreducibles. If you have $V = \oplus n_i V_i$, then $(\chi_V, \chi_{V_i}) = n_i$. This tells you the multiplicity from the character. So what's the catch? It's not a very easy manipulation to do. Even in the rare cases when you can describe the measure, integration is no easy thing. It's rather difficult combinatorics to do this for, say, $SU(n)$. I don't want to integrate over a compact group.

There are cases where you can do this, like for a finite group, where this is a finite sum. In general this is a nice answer in theory but not in practice.

Let me just finish by saying the following. There is a question about whether you can write these functions in a basis-independent way. The other question is whether you get "all" functions on the group.

Basis-independent description of matrix elements:

First of all, what is g_{ij}^V? It is $\langle ge_j, e^i\rangle$. So instead of considering elements of the first kind, consider $\langle gv, v^*\rangle$, for $v \in V, v^* \in V^*$.

Theorem 3 *The map $\bigoplus_V V \times V^* \to C^\infty(G)$ which takes $v \otimes v^*$ to $\langle v, v^*\rangle$.*

This is a coordinate independent way of describing the space generated by my elements. The map has the following properties:

1. *It preserves $\langle , \rangle$. What is the inner product we're using on the tensor product? Choose an inner product on V. Then it defines a measure invariant inner product on V^*. So you define the inner product the only way you can, as*

$$\langle v_1 \otimes v_1^*, v_2 \otimes v_2^*\rangle = \frac{1}{dim\ V}\langle v_1, v_2\rangle\langle v_1^*, v_2^*\rangle.$$

   So the statement is that this would preserve the inner product. And of course the summands are orthogonal.

2. *We didn't say something before when we were talking about the group in terms of matrices.*

   There are actions on the group, including left, right, and adjoint multiplication.

   We have obvious actions on the tensor. So we can take the action of g on both, or only on one factor of the product. So there are several possibilities. There is an exact correspondence between these choices and the actions on $C^\infty G$.

   So under this map, the action of G on V corresponds to the action of G on $C^\infty(G)$ by $(gf)(h) = f(hg)$, the action on V^* to the action by $(gf)(h) = f(g^{-1}h)$.
   The action by both corresponds to $(gf)(h) = f(g^{-1}hg)$.

3. *This map is injective. I should direct sum over isomorphism classes of irreducible representations. I am talking just about the algebraic direct sum.*

So let me say a little bit about the proof. Suppose you have some vectors v and v^*. These correspond to the function $f(h) = \langle hv, v^*\rangle$. So if I take $gv \otimes v^*$, what do I get here? I get $f(h) = \langle hgv, v^*\rangle = f(hg)$.

So okay, now we need to move from finite to infinite direct sums. So you just do the norm completion with respect to the inner product norm. Then you do the same thing in $C^\infty(G)$, with the L^2 norm we've been using.

Theorem 4 *Peter-Weyl theorem The map $\bigoplus_V V \otimes V^* \to L^2(G, dg)$ (direct sum in the sense of Hilbert spaces) is an isomorphism.*

That's a relatively hard theorem. The easy part is that the map exists, preserves norm, intertwines with the action of the group, and is injective. But it is hard to see why it's surjective. So if you look at the space of functions on the group as a representation, it splits into a direct sum of finite dimensional irreducibles, each with finite multiplicity. This is unexpected. But it works. I'm not even going to prove this.

Let me give you just one example.

Example 2 *Let $G = S^1 = \mathbb{R}/\mathbb{Z}$ Then the representations are of the form $V_n = \mathbb{C}$ Where $\psi \to e^{2\pi i n \psi}$ for $n \in \mathbb{Z}$. The right hand side is $L_2(S^1)$. The left hand side is the direct sum $\oplus_{n\in\mathbb{Z}} V_n \otimes V_n^*$. If I consider the right action of the group on itself, then I get that this is $\oplus_{n\in\mathbb{Z}} V_n$. So what is the guy who corresponds to V_n? Well, 1 in V_n corresponds to the subspace $\mathbb{C}e^{2\pi n\phi}$. And, of course, L^2 is the direct sum of spaces spanned by functions of this form. So in this example this is the Fourier series, nothing more.*

So that's it. Okay, unfortunately for other groups it's more complicated. We haven't answered the question about irreducibles for any compact group except S^1. We know the characters are orthogonal, but that doesn't help you find them. So how will we find them? We'll get into that next time.