Introduction to Lie Groups and Lie Algebras
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Gabriel C. Drummond-Cole

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Even though all of you did this pretty well, you are doing more work than necessary. You want to show that in the Lie algebra of a compact Lie group there is always an invariant inner product. You don't need to do this from scratch from the Haar measure. Just refer to the theorem we proved in class.

Similarly, in another problem, you had a real Lie algebra and you had to show that the Killing form was negative semidefinite if $\mathfrak{g}$ admits an invariant positive definite form; then with respect to an orthonormal basis the matrix is in $\mathfrak{su}$ so you can use the first part of the problem.

I said we were going to classify representations of semisimple Lie algebras. So let's begin with representations of $\mathfrak{sl}(2, \mathbb{C})$. Of course I mean complex finite dimensional representations.

Recall that we have three commutation relations: $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Our approach will be based on the following: Any action, in particular the action of h, has eigenvalues. So let's find the eigenspaces of h. The key lemma is as follows:

Lemma 1 *If $hv = \lambda v$ (We denote all such v by V_λ) then $ev \in V_{\lambda+2}$ and $fv \in V_{\lambda-2}$.*

You compute $hev = ehv + [h, e]v = \lambda ev + 2ev = (\lambda + 2)ev; hfv = fhv + [h, f]v = \lambda fv - 2fv = (\lambda - 2)fv$.

Corollary 1 *So if V is irreducible then h is diagonalizable; $V = \oplus_\lambda V_\lambda$, $h|_{V_\lambda} = \lambda \cdot id$.*

Let $W = \oplus V_\lambda$, where V_λ is the kernel of $h - \lambda id$. This is a direct sum but this may not be the whole space. But W is stable under the action of $\mathfrak{su}(2)$ so it is the whole space; it cannot be zero because h must have one eigenvalue.

My next goal will be classification of irreducible representations. Suppose V is irreducible. Then write $V = \oplus V_\lambda$. The terminology is that these are called weight subspaces, and elements of V_λ are called vectors of weight λ.

Choose λ with maximal real part; we can always do this because there are finitely many weights. Let $v \in V_\lambda$. Then $ev_\lambda = 0$. The proof is trivial, because of the lemma. Such v aare called highest weight vectors.

This is absolutely standard terminology. Why it is called highest instead of maximal, I don't know.

Okay, so what. The space spanned by v is stable under the action of e and h so we have no choice but to apply f.

Lemma 2 *So define $v^k = \frac{f^k}{k!}v$.*

1. $hv^k = (\lambda - 2k)v^k$ *This is trivial.*

2. $fv^k = (k+1)v^{k+1}$. *This is also trivial.*

3. $ev^k = (\lambda + 1 - k)v^{k-1}$ *for $k > 0$*

Only the last part requires proof.
Let's do this by induction. First let $k = 1$. Then $v^1 = fv$. So what is efv? it is $fev + hv = hv = \lambda v$.

Now for the induction step, we see

$$ev^{k+1} = e\frac{fv^k}{k+1} = \frac{1}{k+1}(fev^k + hv^k) = \frac{1}{k+1}(f(\lambda+1-k)v^{k-1} + (\lambda - 2k)v^k)$$

$$= \frac{1}{k+1}(k(\lambda+1-k) + \lambda - 2k)v^k = \frac{\lambda(k+1) - k(k+1)}{k+1}v^k = (\lambda - k)v^k.$$

So we're almost done; maybe these vectors are linearly dependent. You have to get a zero vector eventually; in a finite dimensional vector space you will have a minimal eigenvalue so that eventually $v^k = 0$. So let k be maximal such that $v^k \neq 0$. What can we say then?

Well, $ev^{k+1} = 0$ but by our identity we have $ev^{k+1} = (\lambda - k)v^k$. Then $\lambda = k \in \mathbb{Z}_+$. Also, $v, v^1, \ldots, v^k$ are a basis in V. First of all, they generate because the subpace they generate is stable under e, f, h. Why are they linearly independent? Because they have different eigenvalues.

Theorem 1 *If V is irreducible, then it has a basis $v, \ldots, v^\lambda$ with the action of $\mathfrak{sl}(2)$ given by the formulas of the lemma, with $\lambda \in \mathbb{Z}_+$.*

So far, we have shown one direction; conversely for any nonnegative integer λ you can construct such a representation. To show it is irreducible, the only candidates are some subspace spanned by some of the v_i. But any such set generates the whole representation under e and f. As a matter of fact, if you recall your homework this is the exact thing you got for the

k symmetric power of $\mathbb{C}^2$. These are unfortunately also called V_k. These are pairwise non-isomorphic because of dimensional considerations. In the physical literature, you do not classify by positive integers but by $\lambda/2$, called spin. There are good reasons for the standard representation to be spin one half.

That almost completes the story of $\mathfrak{sl}(2, \mathbb{C})$. How do we classify all of them? If we know that every representation is completely reducible, then every $V \cong \oplus_{k \in \mathbb{Z}_+} n_k V_k$.
As examples, $\mathbb{C} = V_0, ad = V_2$.
How do we know that representations are completely reducible. We could refer to the theorem from last time about semisimple Lie algebras. We could also compute the eigenvalue of the Casimir element; it is $k^2/2 + k$. In particular this has different values on different irreducible representations. Therefore you can use the Casimir element to smooth your representation. Even without using complete reducibility we know that $V = \oplus_k V^{(k)}$ where $V^{(k)}$ is a generalized eigenspace for C with eigenvalue $k^2/2 + k$.

For any representation you can split by the eigenspaces of the Casimir element. The next step is to split the $V^{(k)}$ into a composition series with every factor equivalent to V_k. Then you can rather easily show that it actually splits. So Casimir operator can be used to show irreducibility in this case.

So we have classified representations of $\mathfrak{sl}(2)$. I only sketched the second proof, I didn't give full detail.

So the next goal is generalizing this to other semisimple Lie algebras. The key point here was the basis h, e, f with $[h, e]$ a multiple of e and $[h, f]$ a multiple of f, well really that we had an eigenbasis of $ad\ h$.

Definition 1 $x \in \mathfrak{g}$ *is called semisimple if $ad\ x$ is a semisimple, or equivalently diagonalizable operator.*

Why is this important? If x is semisimple then $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$, with $ad\ x|_{\mathfrak{g}_\lambda} = \lambda id$. So $[x, y] = \lambda y$ for $y \in \mathfrak{g}_\lambda$.

Lemma 3 *Let x be semisimple and V a simple representation of $\mathfrak{g}$.*

1. *If $xv = \lambda v, [x, y] = \alpha y$, then $xyv = (\lambda + \alpha)yv$.*
   $\mathfrak{g}_\alpha V_\lambda \subset V_{\lambda + \alpha}$.

2. *In any irreducible representation, x is diagonalizable.*

The proof is the same as it was for $\mathfrak{sl}(2)$. Take the direct sum of the eigenspaces; this is invariant under the action of the Lie algebra and is nonzero, so it's the whole space.

So this is the starting point. As you might guess, what we will do next time (I don't want to go any further) is to start classification for any Lie algebra. For $\mathfrak{sl}(2)$ it sufficed to consider just h. For a general Lie algebra we'll need to choose several commuting diagonalizable elements. We'll have to talk about joint eigenspaces.

I should also say that the diagonalizable commuting elements are also known as Cartan subalgebra, which all of you have seen. What puts semisimple Lie algebras apart is that they have sufficiently many commuting diagonalizable semisimple elements.