Introduction to Lie Groups and Lie Algebras
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Gabriel C. Drummond-Cole

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To do the last problem, you wanted to show that the quotient was semisimple as a way to show that the ideal I gave was the radical.

Today we'll talk about complete reducibility. The goal is to prove that any representation of a semisimple Lie algebra is completely reducible. The algebra can be real or complex but the representation is assumed to be complex and finite dimensional.

Historically, this was done by looking at complex Lie algebras, and then showing that these were the complexifications of compact Lie algebras, and using the Haar measure to get the result.

However, there is a purely algebraic way. It involves two ingredients; one of them is the Casimir element.

Definition 1 *If $(,)$ is an invariant symmetric bilinear form on $\mathfrak{g}$ with x_i an orthonormal basis, then $C = \sum x_i^2 \in U_{\mathfrak{g}}$ is called the Casimir element.*

This doesn't depend on the choice of orthonormal basis. Here C is the image of $\sum x_i \otimes x_i$, and with respect to any basis you can write it as $\sum v_i \cdot v^i$.

Your homework was to compute that this element is central. I won't repeat that, it's easy.

Theorem 1 *Let V be a representation of $\mathfrak{g}$ such that $ker\ \rho = 0$, i.e., $\rho(x) \neq 0$ for nonzero x. Such a representation is called faithful.*
Then $tr_V(\rho(x), \rho(y))$ is a nondegenerate invariant symmetric form.

To show invariance it's easiest to look at the action Ad of a Lie group G, which is conjugation so does not show up in the trace. The only thing we really have to show is that this is nondegenerate. Let $I = ker\ (,) = \mathfrak{g}^{\perp}$. Then I is an ideal in $\mathfrak{g}$, and because of semisimplicity you can write $\mathfrak{g} = I \oplus \mathfrak{g}'$, where the latter is semisimple.

So I itself is semisimple and $tr_V(\rho(x)\rho(y)) = 0$ for all $x, y \in I$. By one of the forms of the Cartan criterion, then $\rho(I)$ is solvable as a subset of $\mathfrak{gl}(V)$. So each simple factor of I must be taken by ρ to zero, since the image must be simple and solvable. Since $ker\ \rho$ is trivial, this means $I = 0$.

This is then a good way to get nondegenerate invariant forms, which automatically give you Casimir elements.

Let V be such a (faithful) representation and C_V the Casimir element corresponding to $(,) = tr_V(\rho(x), \rho(y))$. If I assume in addition that V is irreducible, then let's see how C_V acts in V. I claim that $C_V|_V$, which we know by the Schur lemma to be a multiple of the identity as a central element, is nonzero, and in fact is $\frac{dim\ \mathfrak{g}}{dim\ V} id$.

If x_i is an orthonormal basis, then $tr_V(\rho(x_i)^2) = 1$ so that $tr(\rho(\sum x_i^2)) = dim\ \mathfrak{g}$.

Corollary 1 *For any irreducible representation V of $\mathfrak{g}$ there exists $C \in Z(U_{\mathfrak{g}})$ such that $C|_V = \lambda\, id$ for $\lambda \neq 0$.*

We need to prove this, since the representation may not be faithful. Write $\mathfrak{g} = \oplus_{i \in J} \mathfrak{g}_i$, these being simple. So look to $I = ker\ \rho$. If the kernel is zero then the representation is faithful and we are done. So what if it's not? This ideal is a direct sum $\oplus_{i \in J'} \mathfrak{g}_i,\ J' \subset J$. Then let $\mathfrak{g}' = \oplus_{i \in J - J'} \mathfrak{g}_i$; then V is a faithful representation of $\mathfrak{g}'$, since $\mathfrak{g} = \mathfrak{g}' \oplus I$. We could take $C = C_V \in U_{\mathfrak{g}'}$.

The problem is that $\mathfrak{g}'$ may be zero, which occurs precisely when $V = \mathbb{C}$. You can show that the Casimir element for the Killing form will act nontrivially in any representation, but that's harder.

So that was the first thing we needed. We're ready to prove the main result of today.

Theorem 2 *Any representation of $\mathfrak{g}$ is completely reducible.*

The proof uses a little bit of homological algebra. If you're not familiar with this, then you may not get it all, but it is supposed to be in the core courses.

So we can talk about long and short exact sequences of modules over a ring, and you know that there is a way of classifying all short exact sequences extending V and U to $0 \to V \to W \to U \to 0$. That is, isomorphism classes of W are in bijection with $Ext^1(U, V)$.

The way is as follows. We need to construct an element of $Hom(U, W)$. We have the sequence $0 \to Hom(U, V) \to Hom(U, W) \to Hom(U, U)$ by the left exact functor $Hom(U, \cdot)$. We need to check whether the identity map from U to U can be lifted to a map to W agreeing with the projection. This would be so if $Hom(U, W) \to Hom(U, U)$ were surjective. But we know that this can be extended to a long exact sequence whose next element is $Ext^1(U, V)$. In particular if $Ext^1(U, V) = 0$ then id can be lifted and we are done.

So $Ext^1(U,V)$ is what we need to study. I won't remind you how to define it, because it would take quite a long time.

This argument can be repeated word for word over any abelian category of enhanced modules, including the category of Lie algebras. Or you can use a shortcut and consider *Ext* over universal enveloping algebras.

We need to show that $Ext^1(U,V) = 0$.

Lemma 1 $Ext^1(\mathbb{C},V) = H^1(\mathfrak{g},V) = 0$ *for any irreducible representation* V*.*

This is still a very special case. Let's prove the lemma first. The proof is not very hard.

1. Say that V is not the trivial representation. Then the corollary gives us all we need. We know that *Ext* classifies all short exact sequences of the form $0 \to V \to W \to \mathbb{C} \to 0$. Take $C \in Z(U_{\mathfrak{g}})$ as in the corollary. In V it acts by zero; in $\mathbb{C}$ it acts trivially; how does it act in W? So $C|_V = \lambda \neq 0, C|_{\mathbb{C}} = 0$, so we can decompose W as the direct sum of eigenspaces $V \oplus V'$, where V' is the kernel of the Casimir element. Since it commutes with the action of the Lie algebra, we've found that W splits as a direct sum of representations so there is only one isomorphism class of such W.
2. Say $V = \mathbb{C}$, the trivial representation. We're talking about $0 \to \mathbb{C} \to W \to \mathbb{C} \to 0$, so you have a trivial subrepresentation and a trivial quotient. So there's a basis of W such that every element has the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, since it kills the commutator of any element with the first $\mathbb{C}$, and similar considerations for the quotient.
Can you have a two dimensional irreducible representation of this form? No, because this is nilpotent.

From now on it's rather trivial homological algebra.

Lemma 2 $Ext^1(\mathbb{C},V) = H^1(\mathfrak{g},V) = 0$ *for any* V*.*

The proof is by induction on *dim* V. Either V is irreducible and we are done, or it has a subrepresentation and quotient representation $0 \to V_1 \to V \to V_2 \to 0$. Then we have the long exact sequence of *Ext*, which is locally $Ext^1(\mathbb{C},V_1) \to Ext^1(\mathbb{C},V) \to Ext^1(\mathbb{C},V_2)$. So if you know the long exact sequence of *Ext* then you are done, since the first and third of these vanish.

Now the next thing, after the case where the first factor is trivial and the second irreducible and the case where the first is trivial and the second is anything, is

Lemma 3 $Ext^1(V,U) = Ext^1(\mathbb{C},V^* \otimes U)$.

This doesn't usually make sense, but in this case it does.

I'll wave my hands a little. You add Ext to make the sequence of Homs exact. More formally, Ext is the derived functor from Hom. Then $Hom_{\mathfrak{g}}(V,U) = Hom(\mathbb{C}, V^* \otimes U)$ and we're done.

Just as vector spaces, this is clear, since $Hom(V,U) \cong V^* \times U$ and $Hom(\mathbb{C}, X) \cong X$. Now if $\phi : V \to U$ commutes with the action of $\mathfrak{g}$ then $\phi \in V^* \otimes U$ is $\mathfrak{g}$-invariant; this is a trivial exercise. Therefore we know that $Hom_{\mathfrak{g}}(V,U) = (V^* \otimes U)^{\mathfrak{g}} = Hom(\mathbb{C}, V^* \otimes U)$. This last one is more or less immediate.

The middle term is the space of invariants, of all vectors u such that $xu = 0$ for all $x \in \mathfrak{g}$.

From here it's not so, well, this kind of gives you the idea for why you have the same for Ext. Probably I don't want to go any deeper into that. You need to show that the functor of tensor product is exact, but I don't want to go into it.

But now we're done. By this lemma and the last, $Ext^1(V,U) = Ext^1(\mathbb{C}, *)$ for some $*$, which is zero and we're done.

The arguments in the last two lemmas use no information about semisimplicity, so that is used entirely for the Casimir element.

In particular we now know that any representation is completely reducibles; now how do you do it? That's what we'll have to do starting next time. In theory we could use orthogonality of characters for compact groups, but it is too hard to integrate in real life.

Let me make just one more remark. One can also consider more general Ext spaces. In particular, we can define $H^i(\mathfrak{g}, V) = Ext^i(\mathbb{C}, V)$. Then $H^i(\mathfrak{g}, V) = 0$ for all i if V is irreducible and not the trivial representation.

However $H^i(\mathfrak{g}, \mathbb{C})$ may be nonzero for $i \geq 3$. The study of this homology is rather important. If $\mathfrak{g} = Lie(G)$ for a compact connected Lie group G, then $H^i(\mathfrak{g}, \mathbb{C}) = H^i_{DR}(G, \mathbb{C})$. So $H^3(SU(2)) \neq 0$, for instance. But defining this higher Ext takes some effort, and moreso to get these results. But I wanted to at least mention this result. It's not true for noncompact groups, by the way.

Okay, so I think I actually want to stop here. We'll study a lot of examples starting next time. We'll start studying the irreducible representations from the point of view of Lie algebras, forgetting about Lie groups altogether.