Introduction to Lie Groups and Lie Algebras October 21, 2004

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I think we can begin. So let me remind you what we did before. We were looking at semisimple and solvable Lie algebras, and we had shown that $\mathfrak{g}$ is semisimple if and only if the Killing form K is nondegenerate, which occurs if and only if $\mathfrak{g}$ is the direct sum of simple Lie algebras.

Let's get some immediate corollaries and properties of semisimple Lie algebras.

Corollary 1 *1. Say $\mathfrak{g}$ is real. Then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semisimple.*

2. For semisimple $\mathfrak{g}$, we have $Z(\mathfrak{g}) = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

3. Say $\mathfrak{g} = \oplus \mathfrak{g}_i$ is a decomposition into simple algebras. Then an ideal in $\mathfrak{g}$ is of form $\oplus \mathfrak{g}_i$.

4. Any ideal or quotient of a semisimple algebra is semisimple.

1. The same basis will work for the complexification, and then the Killing form takes the same form. Note that the complexification of a simple algebra may not be simple.

2. An ideal of a semisimple algebra cannot be solvable, much less abelian. For the commutant, write the algebra as the sum of simple algebras, and then an ideal must be the sum of some of these. Then it must contain something from each one of these, so is the sum of all of them.

3. This is a strong statement; it is not about isomorphism but equality. The proof goes by induction in the number of summands. Let $I \subset \mathfrak{g}$; then either $I \supset \mathfrak{g}$ or I is contained in the rest of the components.

   Let $p : \mathfrak{g} \to \mathfrak{g}_1$ be projection. If this is 0 then I is in the rest of the simple summands. If the projection is nonzero then look to $[\mathfrak{g}_1, p(I)]$. This is the same thing as $[g_1, I]$. Since I is an ideal this lives in I, but also lives in $\mathfrak{g}_1$. I claim that it must be the whole $\mathfrak{g}_1$ since $\mathfrak{g}_1$ is simple.

4. We know what the ideals are so this follows directly.

Let $\mathfrak{g}$ be any Lie algebra. Then $Der(\mathfrak{g})$ is $\{d : \mathfrak{g} \to \mathfrak{g} | d[x, y] = [dx, y] + [x, dy]\}$.

1. As an exercise, you can show that $Der\ \mathfrak{g}$ is a Lie algebra.
2. Then $Der\ \mathfrak{g}$ is the Lie algebra of the Lie group of automorphisms of $\mathfrak{g}$.

   If I was going to figure out the Lie algebra of the group of automorphisms, you'd parametrize and take derivatives with respect to time, and what you get is the Liebnitz rule.

   It's a Lie group because it is a closed subgroup of $GL(n)$. We have a theorem that tells us that a closed subgroup of a Lie group is a Lie group. We didn't prove it but if we use it then you can easily see that $Der\ \mathfrak{g}$ is the Lie algebra.
3. $ad : \mathfrak{g} \to Der\ \mathfrak{g}$. So this is one of the many forms of the Jacobi identity. $ad_x\ [y, z] = [ad_x\ y, z] + [y, ad_x\ z]$. Further, the image of this is an ideal in $Der\ \mathfrak{g}$.

Why? If $d \in Der\ \mathfrak{g}$ and $x \in \mathfrak{g}$ then all we must show is that $[d, ad_x] = ad_y$ for some y. So let $y = d_x$. This is another form of the Jacobi identity so I don't think I want to write it.

In general derivations of this form are called inner, and the corresponding automorphisms are called inner automorphisms; the others are called outer automorphisms. There are often outer automorphisms, for instance over a commutative Lie algebra.

For semisimple Lie algebras you have a nice piece of information, which is that there are no other derivations.

Theorem 1 *If $\mathfrak{g}$ is semisimple then $ad : \mathfrak{g} \to Der\ \mathfrak{g}$ is an isomorphism.*

Let me note that this is injective, because the kernel of ad is those elements for which $ad_x = 0$, i.e., central elements. But semisimple algebras have no center.

The hard part is surjectivity. Define a bilinear form K on $Der\ \mathfrak{g}$ by $K(d_1, d_2) = tr_{\mathfrak{g}}(d_1 d_2)$. This is just the natural extension of the Killing form on $\mathfrak{g}$ to a possibly larger algebra $Der\ \mathfrak{g}$. Trace is always an invariant symmetric bilinear form. Consider $\mathfrak{g}$ and its orthogonal complement in $Der\ \mathfrak{g}$. Since the form is invariant and $\mathfrak{g}$ is an ideal, it is a trivial exercise that $\mathfrak{g}^{\perp}$ is an ideal. What prevents me from doing the same for any Lie algebra? For now, nothing.

But now I claim that $\mathfrak{g} \cap \mathfrak{g}^{\perp} = 0$. The intersection would be the kernel of the Killing form on the Lie algebra, but the Killing form is nondegenerate on a semisimple algebra.

So $Der\ \mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$. If $d \in \mathfrak{g}^{\perp}$ then $[d, ad_x] = 0$, but that happens exactly when $ad_{dx} = 0$. So $dx = 0$ for all x, so that $d = 0$.

Okay, so we are done. As a result we get that any derivation is inner for a semisimple Lie algebra. At the level of Lie algebra automorphisms we don't quite have it.

Corollary 2 *If G is a connected Lie group with semisimple Lie algebra (such a group is called semisimple) then $(Aut\ \mathfrak{g})^0 = G/Z(G)$.*

That is, this only tells you about the connected component of the identity. There are possibly some finite number of outer automorphisms which you can product with $G/Z(G)$ to get other automorphisms.

Example 1 *For $SL(n, \mathbb{C},\ Aut(\mathfrak{g})$ has two connected components. For $SL(2)$ it takes A to PAP where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So this sends e and f to one another and h to $-h$. It is not obvious but this is not an inner automorphism. Other than this discrete set there is nothing else.*

Let me go back to another thing. If the Lie algebra is real then you can ask whether the Killing form is positive definite, negative definite, or what.

Theorem 2
1. *If $\mathfrak{g}$ is a real semisimple Lie algebra with negative definite Killing form, then $\mathfrak{g} = Lie(G)$ for some compact Lie group.*
2. *If G is a compact Lie group then $\mathfrak{g} = Lie(G)$ is isomorphic to the direct sum of an Abelian Lie algebra $\mathfrak{a}$ with a semisimple Lie algebra $\mathfrak{s}$ with negative definite Killing form*
3. *The only real semisimple algebra with positive definite Killing form is 0.*

Corollary 3
1. *$\mathfrak{so}(n, \mathbb{R}), \mathfrak{su}(n)$ are semisimple, and $\mathfrak{u}(n)$ is reductive, i.e., $a \oplus s$.*
2. *$\mathfrak{sl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$ are semisimple, and $\mathfrak{gl}(n, \mathbb{R})$ is reductive.*

Look at $SO(n, \mathbb{R})$. It is compact so its Lie algebra is reductive. By the Schur lemma, a matrix in $\mathfrak{so}(n, \mathbb{R})$ which commutes with everything is a multiple of the identity, so no such exist.

The same argument works for $\mathfrak{su}$. So what is the complexification of $\mathfrak{su}(n)$? It is $\mathfrak{sl}(n, \mathbb{C})$, so this is also semisimple. This also gives semisimplicity of $\mathfrak{sl}(n, \mathbb{R})$. You could also explicitly write the Killing form and show it to be nondegenerate.

The symplectic Lie algebra is also semisimple, by a similar trick. It can also be seen as the complexification of a Lie algebra of a compact group.

So we haven't yet proved this. Let me prove it and that will probably be the last thing I do today.

1. Consider $\mathfrak{g}$ and the form $-K$, which is a positive definite form. Look at $Aut(\mathfrak{g})$. I claim that every automorphism of $\mathfrak{g}$ will preserve the Killing form. That makes sense because

it preserves the commutator. Then of course it also preserves $-K$. Then $Aut(\mathfrak{g})$ is a subgroup of $O(\mathfrak{g})$. This means the group which preserves $-K$.

By a choice of basis this is isomorphic to $O(n, \mathbb{R})$ where $n = dim\ \mathfrak{g}$. So this group of automorphisms is a closed subgroup of O. So $Aut(\mathfrak{g})$ is compact. Then every connected component is compact. Then $\mathfrak{g}$ is the Lie algebra of the connected component of the identity, $\mathfrak{g} = Lie((Aut(\mathfrak{g}))^0)$. I could not have said that the adjoint action defines a group, because the image may not be closed. To avoid this problem I deal with the group of automorphisms, which is closed because it's defined by an infinite number of closed conditions.

2. This is part of the homework. I gave you discrete center which kills a so I won't do that one.

3. You can actually get this by combining the first two. The same argument gives that $Aut(\mathfrak{g}) \subset O(\mathfrak{g}, K)$. So $\mathfrak{g}$ is the Lie algebra of a compact group. Whether or not the original form was positive or negative you get the same answer. But then the second part tells you that the Killing form is negative definite. It must also have a negative definite Killing form. You can only have a form which is both positive and negative definite unless your space is zero.

This is a very important theorem because we know that compact groups are nice, and now we can decompose real semisimple Lie algebras, and, it turns out, complex ones, although it is trickier. The important part is that they have nice representation theory, which doesn't quite work because we need the Lie algebras to have a negative definite Killing form.

We don't know yet that we can choose the basis to make the Killing form negative definite, but you can do this for any complex semisimple Lie algebra.

Theorem 3 *(H Weyl)*

1. *Any complex semisimple Lie algebra $\mathfrak{g}$ can be written as $\mathfrak{g} = (\mathfrak{k})_{\mathbb{C}}$ for some real Lie algebra $\mathfrak{k}$ with compact Lie group.*
2. *Any finite dimensional representation of a complex Lie algebra is completely reducible.*

This is the theorem. I'm not proving it. It is often called the unitary trick, because every compact Lie group representation is unitary.

Here is the statement, and to be honest I don't really know how to prove the first part. You can find it in plenty of places. What we will do next time is prove the second part without using the first part at all. After all, the statement is completely algebraic. Why on earth should you have to use analysis, involving things like invariant measures. There is an algebraic proof which was found later, and that's what I'll do next time.

It will use Ext and long exact sequences so if you are not very familiar with those you should look them up.