Introduction to Lie Groups and Lie Algebras
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For the homework, I have some comments. If V is irreducible, then dualization reverses inclusion and quotient, so that if $U \subset V^*$ then $U^* = V/U^\perp$. There is a standard way of doing this without a basis. Only a very little changes when you consider these as representations.

There were a number of minor things with the problem about diagonalizing the operator on the cube. If $Z : V \to V$ and $Z^2 = 1$ then it is almost immediate that $V = V_+ \oplus V_-$. You don't need a basis to get these. I thought that this trick is so familiar that all of you know. What you need to check is that each of these is a representation, so that they are stable under the action of $\mathfrak{g}$. It follows from the fact that Z commutes with the action of the group. This implies that the eigenspaces of Z are representations.

Doing this with bases obscures the real reasons that things work; it's not formally wrong but you should try to avoid using bases if there's a way not to. Maybe I should say something about the last problem. The problem was with the first part, extending the action to $\mathbb{R}^4$. The standard inner product is $\frac{1}{2}tr(x\bar{y}^t)$, you can show this. I made a mistake with the coefficient. Then the action can be by left multiplication. You need to check that gA is in the same subspace; then it is immediate that this map is a linear action of your group on a vector space, and it is equally trivial to show that it preserves the bilinear form. It is harder to show linearity by extending multiplicatively; you don't know if its extension is linear.

Okay, fine. Maybe at some later time I'll actually post the solutions. Now let's move on and continue with what we're doing right now. We defined a semisimple Lie algebra as one containing no solvable ideals.

It followed directly that a Lie algebra is made out of a solvable ideal, the radical, and a semisimple one, the quotient by this radical. I made a slightly more complicated statement, that $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_{ss}$. The first is an ideal, the second is a subalgebra.

We talked about how to tell if a Lie algebra is semisimple.
Cartan Criterion

1. $\mathfrak{g}$ is solvable if and only if $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

2. $\mathfrak{g}$ is semisimple if and only if K is nondegenerate,

where $K(x, y) = tr_{\mathfrak{g}}(ad\ x, ad\ y)$ is the Killing form.

Today I'm going to prove this, so I need some things about linear algebra. Let's forget about Lie algebras for a second and talk about linear algebra on a complex vector space.

Definition 1 *Let V be a finite-dimensional complex vector space. Then $A : V \to V$ is nilpotent if $A^n = 0$ for $n >> 0$ and semisimple if A is diagonalizable.*

Theorem 1 *Any $A \in End(V)$ can be uniquely written as $A = A_s + A_n$, where A_s is semisimple and A_n is nilpotent, with $A_s A_n = A_n A_s$.*
Moreover, $A_s \in A\mathbb{C}[A], A_n \in \mathbb{C}[A]$.

The proof is rather simple. Write $V = \bigoplus V_\lambda$. Here V_λ is the kernel of $(A - \lambda I)^n$ for $n >> 0$. Let $A_s|_{V_\lambda} = \lambda id$ and $A_n = A - A_s$. This is obviously nilpotent because of the kernel condition; they commute because A_s commutes with anything. Let $p \in \mathbb{C}[x]$ be a polynomial such that $p \equiv \lambda_i \mod (x - \lambda_i)^{d_i}, p \equiv 0 \mod x$. Equivalently $p = \lambda_i$ in $\mathbb{C}[x]/(x - \lambda_i)^{d_i}$. I laim that this system of congruences always have a solution. This exists by the Chinese Remainder Theorem.

Then $A_s = p(A), A_n = A - p(A)$. For uniqueness, if $A = A'_n + A'_s$ then $A_n + A_s = A'_n + A'_s$ so $(A_n - A'_n) = (A'_s - A_s)$. Then A'_n commutes with A and so with A_n, a polynomial in A_n. Well, the difference of commuting nilpotent operators in nilpotent; the difference of commuting semisimple operators is semisimple. So you have a nilpotent operator equal to a semisimple operator. It is diagonalizable but all eigenvalues are 0 so it is 0.

This was the hardest one. Let me give you a second one.

Theorem 2 *Let $ad\ A : End(V) \to End(V)$ be the adjoint action by A. Then we can discuss its eigenvalues and try to decompose it into the sum of semisimple and nilpotent operators. Then $(ad\ A)_s = adA_s$. This has eigenvalues $\lambda_i - \lambda_j$, for $\lambda_{i,j}$ eigenvalues of A.*

The proof is as follows. If you choose a basis in V which is an eigenbasis for the semisimple part, and such that the nilpotent operator is upper triangular, which you can do (essentially this is one of the steps in the proof of Jordan normal form), then $ad\ A_s$ is diagonal in the basis E_{ij} of $End(V)$ and $ad\ A_n$ is upper-triangular. So $A_s E_{ij} - E_{ij} A_s = (\lambda_i - \lambda_j) E_{ij}$.

So as soon as we have chosen an appropriate basis, *ad* of the semisimple part is semisimple. For nilpotence you have to work a little harder, think about the order you put on pairs, but it's not very hard.

So A can be written as the sum of a nilpotent and an upper-triangular.

Corollary 1 $(ad\ A)_s = p(ad\ A), p \in x\mathbb{C}[x]$.

So basically what we are doing so far is decomposing into semisimple and nilpotent.

Theorem 3 *Let $\bar{A}_s = \bar{\lambda}_i$ on V_{λ_i} if $A_s = \lambda_i$ on V_{λ_i}. For semisimple operators there are no choices involved. Then we can define $ad\ \bar{A}_s$ and this is $Q(ad\ A)$ for some $Q \in x\mathbb{C}[x]$.*

That's not immediately obvious but here it is. As a proof, $ad\ \bar{A}_s$ has eigenvalues $\lambda_i - \lambda_j$. On the other hand, we can always find a polynomial such that $f(\lambda_i - \lambda_j) = \bar{\lambda}_i - \bar{\lambda}_j$. If you only have finitely many values then you can write conjugation as a polynomial, since you can circumscribe finitely many values of a polynomial. Then $ad\ \bar{A}_s = f(ad\ A_s) = f(p(ad\ A)$.

Maybe I should continue with one more theorem about linear operators.

Theorem 4 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$, such that $tr(x[y,z]) = 0$ for all $x, y, z \in \mathfrak{g}$. Then any $x \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

For a proof, it is very easy. If $x = [y, z]$ then we want to show that all eigenvalues are 0. Compute the following thing: $tr(\bar{x}_s\ x)$. This will be $\sum \lambda_i \bar{\lambda}_i = \sum |\lambda_i|^2$ but this is also $tr(\bar{x}_s, [y, z]) = tr([\bar{x}_s, y]z = tr((ad\ \bar{x}_s\ y)z) = tr(\sum a_k(ad^k x\ y)z)$. I replace $ad\ \bar{x}_s$ with a polynomial in $ad\ x$.

It is a somewhat long proof, but no one knows any better one. This decomposition will be useful again later. Now we can prove Cartan's criterion; it will take us all of five minutes.

For part 1, if $\mathfrak{g}$ is solvable then the Killing form vanishes. Choose a basis in $\mathfrak{g}$ such that $ad\ x$ is upper triangular. Then $ad\ [y, z] = [ad\ y, ad\ z]$ is strictly upper triangular so $tr(ad\ x[ad\ y, ad\ z]) = 0$ since the product of an upper triangular and a strictly upper triangular operator is strictly upper triangular.

Assume the Killing form is zero. You have a vector space, $\mathfrak{g}$, and a subalgebra such that the condition of the last theorem holds. Then by that theorem, $ad\ x$ is nilpotent for $x \in [\mathfrak{g}, \mathfrak{g}]$. Then we can use the theorem that tells us from this that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, and the theorem that tells us from that that $\mathfrak{g}$ is solvable.

Now the second part will be straightforward. If it is semisimple, look at the kernel b of the Killing form When you have an invariant bilinear form, so the kernel is an ideal. What can we say about the Killing form restricted to b. It's 0 Therefore the ideal is solvable. If a Lie algebra is semisimple then it has no solvable ideals except 0, so the kernel is 0 and the form is nondegenerate.

For the other direction, assume K is nondegenerate. Then if $b \subset \mathfrak{g}$ is a solvable ideal, then the commutant sequence terminates. Now, each commutant in the sequence is an ideal in $\mathfrak{g}$. What will be the last one? At some step, you'll get b^n to 0, so that b^n is a commutative ideal. Call this commutative ideal by a.

Now $ad\ x$ is block upper triangular and $ad\ y$ is strictly block-upper triangular for $y \in a$, where the upper blocks correspond to a Their product is strictly upper triangular so $tr_{\mathfrak{g}}(ad\ x\ ad\ y) = 0$.

I cheated in the last part because $K(b, b)$ may be different for $\mathfrak{g}$ and for b. You take the trace acting in different places. It is not hard to see that they agree.

So we're done. It took us a lot of effort, but in the last four minutes we get the main result.

Theorem 5 *A Lie algebra is semisimple if and only if it is isomorphic to a direct sum of simple ones.*

The proof in one direction isn't worth much discussion. If it's a direct sum of simple algebras, then the ideals are sums of these, and there are no solvable ones.

Suppose you have a semisimple Lie algebra. How do you show it is a direct sum of simple ones. Let's argue by induction on dimension. If the Lie algebra is simple, then we are done. Otherwise, it has an ideal I. We can take $I^\perp$ with respect to the Killing form. This is also an ideal, but since the form is not positive definite we can't immediately decompose into direct sums. But let's look at the intersection. $I \cap I^\perp$ is an ideal, and what can we say about the Killing form restricted to this intersection? It's identically 0. So this intersection is solvable. Since the Lie algebra is semisimple then the intersection is 0 so this is a direct sum. Repeat this process.