Introduction to Lie Groups and Lie Algebras
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Recall solvable and nilpotent Lie algebras. Remember that we showed that given a choice of basis a representation of a solvable algebra is upper triangular.

Theorem 1 *(Engol) Let V be a representation of a Lie algebra $\mathfrak{g}$ such that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$. Then there is a basis in V such that all $\rho(x)$ are strictly upper triangular.*

This theorem I'm not proving. The proof is the same as the one done last time with the flags, by induction. It's not particularly difficult but it is not enlightening. It doesn't make much sense to go over this kind of proof again.

Let me just say that there are a number of corollaries of this theorem.

Corollary 1
1. *In this situation $\rho(\mathfrak{g})$ is nilpotent as a subalgebra of $\mathfrak{gl}(V)$, since it is a subalgebra of the strictly upper triangular matrices.*
2. *$\mathfrak{g}$ is nilpotent if and only if ad x is nilpotent for all x.*

To prove one way is obvious. If the Lie algebra is nilpotent, then any commutator you can write with sufficiently many terms is 0, so then if you put the same x enough times then $(ad\ x)^n = 0$.

For the other direction it's not trivial. You only use a little bit of the result. Let $ad\ x$ be nilpotent for all x. Then $ad\ \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is nilpotent by the Engol theorem. So
$[ad\ x_1, [\ ad\ x_2, \cdots, ad\ x_n] \cdots] = 0$ so $ad[x_1 \cdots [x_{n-1}, x_n] \cdots] = 0$ so $[y, [x_1, \cdots x_n] \cdots]] = 0$.

This is a very useful criterion.
So what can we say about general Lie algebras? Do you know structure theorems of associative algebras? You can split into something solvable and something as close to solvable as possible.

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra. Then the radical $\mathfrak{r}(\mathfrak{g}) \subset \mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$, meaning that every solvable ideal in $\mathfrak{g}$ is contained in $\mathfrak{r}(\mathfrak{g})$.*

So the first question is whether these exist, but there is no problem. You don't even need Zorn's lemma because these are finite dimensional. If you have two solvable ideals I_1, I_2, then there is a solvable ideal which contains both of them. Take $I_1 + I_2$. This is obviously an ideal. But $0 \to I_1 \to I_1 + I_2 \to I_2/(I_1 \cap I_2) \to 0$ is exact. Then I_1 and the quotient of I_2 are solvable, so the the center ideal is solvable, so it too is solvable.

So what? So we defined the radical. Now comes the second definition.

Definition 2 *A Lie algebra is semisimple if* $\mathfrak{r}(\mathfrak{g}) = 0$.

If you're familiar with the theory of associative algebras, it's the same in that case.

Example 1 *If a Lie algebra is simple (i.e., has no nontrivial ideals and is not one-dimensional) then it is semisimple. There are good reasons to exclude one-dimensional ideals, such as that their inclusion would make this statement false.*

So $\mathfrak{r}(\mathfrak{g})$ *is either* 0 *or* $\mathfrak{g}$*. How do we know that it is the first choice? The second means* $\mathfrak{g}$ *is solvable, but the commutant is* 0 *so such an algebra must be commutative. Only the one dimensional one can be simple.*

There are other reasons too. Commutative Lie algebras and $\mathbb{R}$ *in particular don't really look like simple algebras.*

Example 2 *Consider* $\mathfrak{g} = \mathfrak{sl}(2)$, *either real or complex. This is simple.*

How do we check. It's not one-dimensional, so that part is done. We need to show that it has no ideals. What are its ideals? If $I \subset \mathfrak{g}$ we must have $ad\ hI \subset I$ for the operator h that we have used in the homework. But in the basis e, f, h this has matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So this is an operator with distinct eigenvalues operating on a vector space. So which are the invariant subspaces?

Lemma 1 *If* $A : V \to V$ *is diagonalizable and has distinct eigenvalues, then* $W \subset V$ *with* $AW \subset W$ *we have* $W = \langle v_i \rangle_{i \in I}$ *(these are the eigenvectors) for* $I \subset \{1, \dots, n\}$.

So if I apply it to this case the possibilities are that such an ideal should be generated by taking some of these three elements, if it exists. There are eight possibilities. It is very easy to show that the only real possibilities are if you take all or none of them. If you have e, then because $[e, f] = h$ you have h in the ideal. Then you also have $f = \frac{1}{2}[h, e]$. Clearly the same argument shows that if it contains f or h it contains the others. So you have no choices at all, other than 0 and the whole space.

So this is simple and thus semisimple. This was a low-tech way, i.e., by hand. You can use the same argument, remarkably, to show that $\mathfrak{sl}(n)$ is also simple. In fact,
$\mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{su}(n), \mathfrak{sp}(2n)$ are all simple except for $\mathfrak{so}(4)$ which is semisimple.

Theorem 2 *If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple.*

So every Lie algebra is built out of a semisimple and solvable one.

We need to show that there are no solvable ideals in the quotient. if I is solvable ideal in $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$, then $\pi^{-1}(I)$ is a solvable ideal in $\mathfrak{g}$, where π is the quotient map.

This is true because it projects under $0 \to \mathfrak{r}(\mathfrak{g}) \to \pi^{-1}(I) \to I \to 0$ and the kernel and cokernel are solvable. This is a solvable ideal larger than the radical, contradicting the definition. So we see that any Lie algebra can be written in the form $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}_{ss} \to 0$.

Theorem 3 *Levi*
You can actually lift back so that $\mathfrak{g}_{ss}$ is a subalgebra in $\mathfrak{g}$, so you get a semidirect product. $\mathfrak{g} = \mathfrak{g}_{ss} \overset{+}{\ltimes} \mathfrak{r}$.

Example 3

1. *For $\mathfrak{g} = \mathfrak{gl}(n)$ you get $\mathfrak{sl}(n) \oplus \mathbb{C}I$.*
2. *Here is a more interesting example. The Poincaré group $G = \{\varphi : \mathbb{R}^n \to \mathbb{R}^n$ wit $\varphi(x) = Ax + B$ with $A \in SO(n, \mathbb{R})$ and $B \in \mathbb{R}^n$.*

   Then $G = SO(n, \mathbb{R}) \ltimes \mathbb{R}^n$. So $\mathbb{R}^n$ is normal in this, but $SO(n, \mathbb{R})$ is not. So you quotient by $\mathbb{R}^n$ and can get $SO(n)$.

   At the level of Lie algebras you get the same thing
   $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, with $\mathbb{R}^n$ under the trivial bracket. Then $\mathbb{R}^n$ is solvable and $\mathfrak{so}(n)$, though I haven't proven it, is simple. So this is a much more typical example.

So this is very nice but we don't have tools to check radicals or see if things are solvable or semisimple. Computing commutants is not easy to do. We can do radicals by hand like we did for $\mathfrak{sl}(2)$ but that's not very nice, so is there another way to do this?

Recall invariant bilinear forms. This is just a map $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ which satisfies the Liebnitz rule $B(ad\ x\ y, z) + B(y, ad\ x\ z) = 0$ This is equivalent to checking that the bilinear form is Ad-invariant. Why are these of use?

Lemma 2 *If $I \subset \mathfrak{g}$ is an ideal, then $I^\perp$ is also an ideal.*

Say $a \in I^\perp$. We want to check that $ad\ x\ a$ is in $I^\perp$. So we need to see what is $B(ad\ x\ a, i)$ for $i \in I$. But this is $-B(a, ad\ x\ i) = 0$ since I is an ideal. Therefore $a \in I^\perp$. That's the same as showing that orthogonal complements in a vector space under a group representation are invariant.

We don't have positive definiteness over $\mathbb{C}$, but forget that for now. It would be good if we had some nondegenerate bilinear forms.

Example 4 *Let V be a representation of $\mathfrak{g}$. Then $B(x, y) = tr_V(\rho(x)\rho(y))$. This is an invariant form.*

We need to check that $tr_V([x,y]z + y[x,z]) = 0$. So I should strictly speaking write $\rho(x)$ so I am being sloppy. This is

$$tr_V(xyz - yxz + yxz - yzx) = tr(xyz) - tr(yzx) = 0.$$

If your Lie algebra comes from the Lie group then you are checking that the form is invariant under the group action. But this proof requires no knowledge of Lie groups at all.

Note that I haven't said anything about degeneracy. In particular, there is a standard representation that exists for any Lie algebra, well two. There is one nontrivial way of constructing a representation, and that is the adjoint representation.

Definition 3 *$K(x,y) = tr_{\mathfrak{g}}(ad\ x, ad\ y)$ is the Killing form.*

It should properly be called the Cartan Killing form, but whatever. It does give us a bilinear invariant form on any Lie algebra.

Here is an example. Let $\mathfrak{g} = \mathfrak{sl}(2)$. Then in the basis e, f, h, then

$$ad\ h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$ad\ e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$ad\ f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}.$$

This doesn't look particularly nice, but what is $K(h,h)$? It is 8. We have $K(e,f) = 4$, if you do the math, and also $K(f,e)$. I forgot to mention that this form is symmetric, which is trivial because $tr(AB) = tr(BA)$. These are the only ones; the others are all 0. So it turns out that it is quite constructive to compare this to the known form from $tr_{\mathbb{C}^2}$. We have $tr_{\mathbb{C}^2}(h^2) = 2$ and $tr_{\mathbb{C}^2}(ef) = 1$, so that this is a multiple of the Killing form.

[Why do we use e, f, h?]

This is tradition. You could do something different but everyone would look funny at you.

This is not surprising; if you have an irreducible representation then there is only one bilinear form up to a constant. It is no surprise that these two gave the same result.

Theorem 4 *Cartan criteria*
Let $\mathfrak{g}$ be a Lie algebra, K the Killing form.

1. *$\mathfrak{g}$ is solvable if and only if $K(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$*

2. $\mathfrak{g}$ *is semisimple if and only if* K *is nondegenerate.*

This helps to find semisimple things, but it is still hard since you have to work in n^2 variables. It's still orders of magnitude better than trying something else.

The proof will come next time.