Introduction to Lie Groups and Lie Algebras
October 11, 2004

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November 30, 2004

About the homework, here is the next assignment. I do ask that you do the homeworks because that's the only real way of learning something. Anything we need to discuss from the previous homework? Anything you wanted to discuss?

I could give you some advice, but you won't like it. It is, you'd better start doing it before the last day before it's due, but some time in advance. I told you you wouldn't like it. You seem to be the only one who has his homework here.

We've been talking about representations and the universal enveloping algebra. Unfortunately, all of it is very nice but it doesn't give us anything to help us classify all representations of a Lie algebra. The way we do it is forgetting completelyh about Lie groups and talking only about Lie algebras. We start with a structure theory of Lie algebra and proceed from there to classifying representations.

I won't repeat what a Lie algebra is. For a Lie algebra we have the same kind of things as for other algebras and groups. We can define a subalgebra, should I give the definition? I have already given it. You can also get an ideal, which is $[\mathfrak{g}, I] \subset I$, and then you can do a quotient. I don't think I need to tell you how the quotient is defined.

Of course you have the familiar result that if you have some morphism of Lie algebras then the kernel is automatically an ideal and the image can be written as $\mathfrak{g}_1/ker\ \varphi$. I don't think there's very much need to prove this.

Example 1 *The commutant of a Lie algebra* $[\mathfrak{g}, \mathfrak{g}]$ *is spanned by all elements of the form* $[x, y]$ *as a vector space.*

The first claim is that this is an ideal. This is pretty obvious. If I commute two things they're automatically in here.

Secondly, what can I say about the quotient? It's abelian, or commutative, so that the commutator is zero.

Third, it is the smallest ideal with this property.

So how do you know that the quotient is abelian? Any commutator is in the ideal so is zero in the quotient. If you want a quotient to be abelian then the kernel must contain all the commutators, so this is trivial.

Example 2 *1. If $\mathfrak{g} = \mathbb{R}$ then $[\mathfrak{g}, \mathfrak{g}] = 0$.*

2. If $\mathfrak{g} = \mathfrak{gl}(n)$ then $[\mathfrak{g}, \mathfrak{g}] = \{x | tr(x) = 0\} = \mathfrak{sl}(n)$.
Then $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = \mathfrak{gl}(n)/\mathfrak{sl}(n)$ is a one dimensional space. It is exactly the trace.

It is clear that a commutator should be traceless, but it requires a little more work to express any traceless matrix as a commutator. You do this for the matrices $E_{ij}, i \neq j$ and $E_{ii} - E_{jj}$, which generate the traceless matrices.

We want a rough classification of Lie algebras. We can't do it exactly, but we want to split it into something almost commutative and something very far from commutative.

The point is that we will classify Lie algebras according to their distance from commutativity. One way to make this precise is with the commutant. The larger the commutant, the further away it is from being abelian.

Definition 1 *Let $\mathfrak{g}$ be a Lie algebra. Define $D^i\mathfrak{g}$ by $D^0\mathfrak{g} = \mathfrak{g}$, $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$.*

We say that the Lie algebra is solvable if $D^i\mathfrak{g} = 0$ for $i >> 0$. This might not look like the definition of solveable for groups or associative algebras. An equivalent thing is that if you have sufficiently nested commutators, this vanishes.

So if you like you can express these as binary trees since at each level you are combining two identical trees.

There is another example which may seem more natural. Consider the following:

Definition 2 *$D_0\mathfrak{g}$ is $\mathfrak{g}$ and $D_{i+1}\mathfrak{g} = [\mathfrak{g}, D_i\mathfrak{g}]$.*
We say $\mathfrak{g}$ is nilpotent if $D_i\mathfrak{g} = 0$ for $i >> 0$.

This is the most general definition; anything you can get using n commutators is in here.

So let's discuss properties.

1. nilpotent implies solvable.
2. a Lie algebra is solvable if and only if there exists a sequence of subalgebras $0 \subset I_0 \subset \cdots \subset I_n = \mathfrak{g}$, each of which is an ideal of the next one, and the quotient of I_j by I_{j-1} is abelian.

   So if you like you're getting your algebra by extension, and each time by a commutative algebra.

3. Nilpotent and solvable are closed under taking subalgebras and quotients. So if $\mathfrak{g}$ is nilpotent (solvable) then $\mathfrak{h} \subset \mathfrak{g}$ or $\mathfrak{g}/I$ is nilpotent (solvable).

4. Suppose you have a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}' \to 0,$$

and that $\mathfrak{h}$ and $\mathfrak{g}'$ are solvable. Then $\mathfrak{g}$ is solvable.

So anything you can get from solvable algebras is solvable. How do we prove these properties?

1. $D_i \supset D^i$, since the first is $[\mathfrak{g}, D_{i-1}]$, and the second is $[D^{i-1}, D^{i-1}]$. This works by induction. Saying it is solvable says that any tree of a certain form is 0; saying it is nilpotent says that any tree of a certain length is 0.

2. Say $\mathfrak{g}$ is solvable. Then $D \subset \cdots \subset D^2 \subset D^1 \subset \mathfrak{g}$ is a sequence as in the condition. Look at the other direction. Say you have the right kind of sequence. Then $I_{n-1} \supset [\mathfrak{g}, \mathfrak{g}]$. So the nth commutant lives in I^{n-n}.

3. The same thing works for subalgebras and quotients obviously.

4. Let me leave it to you to find an appropriate route.

Do we have any good examples of solvable and nilpotent algebras?

Example 3 *1. Heisenberg Lie algebra $\langle x, y, z\rangle$ with $[x, y] = z$ and $[z, \cdot] = 0$. You can check that this algebra satisfies the Jacobi identity. The commutant is $\langle z\rangle$. This is obviously solvable and actually nilpotent.*

2. b is the Lie algebra of upper triangular matrices and n the strictly upper triangular matrices. So the first one is solvable and the second one is nilpotent.

If you remember the notion of a flag, more generally, if you have $\mathscr{F} = 0 \subset V_1 \subset \cdots \subset V_n = V$ with dimension of V_i equal to i, this is what you need to define $b(\mathscr{F}) = \{x : xV_i \subset V_i\}$ and $n(\mathscr{F}) = \{x : xV_i \subset xV_{i-1}\}$. So this corresponds exactly to upper triangular and strictly upper triangular matrices.

The claim is that $b(\mathscr{F})$ is solvable and $n(\mathscr{F})$ is nilpotent. So what is the commutant of b with itself? This is $[b, b] \subset \{x : xV_1 \subset V_{i-1}\}$. What happens is that the diagonal element is destroyed. So then $b_k \subset \{x : xV_i \subset V_{i-k}\}$ so that these commutators converge to 0. The same argument shows why the other algebra is nilpotent. Each time you shift by one degree. So this shows that the inclusion of solvable into nilpotent is strict.

Theorem 1 *Let $\mathfrak{g}$ be a solvable Lie algebra and V a complex finite dimensional representation of $\mathfrak{g}$. Then you can choose a basis such that the matrices of $\mathfrak{g}$ can be written as upper triangular matrices. Equivalently, you can say that there exists a flag $\mathscr{F}$ such that $\rho(g) \subset b(\mathscr{F})$.*

This is a generalization of a known result, which is that you can make a solvable Lie algebra upper triangular.

How do we prove this? Let me explain the basic idea. For one operator we know how to do this. You have two operators which commute, then it is not difficult to modify the proof to find a basis for them both to be upper triangular. We have a solvable Lie algebra which means it is not very noncommutative. So say we have something for several commuting operators, can we adapt it to move up by one?

It suffices to prove the following lemma:

Lemma 1 *There is a common eigenvector for all $\rho(x), x \in \mathfrak{g}$.*

If I have this, then deducing the theorem is rather easy, should I explain how? First I want to prove that the lemma implies the theorem. Suppose I have a common eigenvector (the proof goes by induction in the dimension of V. So there is a common eigenvector v and you can look at $V' = V/\mathbb{C}v$. Then there is a Lie algebra on the quotient so that by the induction assumption there already is a basis in V' so that the action of the Lie algebra is upper triangular. Then denote this basis by $e'_2, \dots, e'_n$.

Then I construct a basis in V by taking this eigenvector and adding to it preimage of each of the e'_i. On the one hand $\rho(x)e'_i$ is a linear combination of e'_j with $j \leq i$ and maybe also some v.

Okay, so that's the trivial part. The difficult part is how do you prove the lemma?

Let's do it by induction on the dimension of $\mathfrak{g}$. Since this is solvable, we know $[\mathfrak{g}, \mathfrak{g}]$ is strictly smaller than $\mathfrak{g}$. So take $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}]$, with codimension of $\mathfrak{h}$ equal to 1. If the dimension of $\mathfrak{g}$ is 1, then $\mathfrak{h}$ can be 0.

So particularly, $\mathfrak{h}$ is an ideal, and $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}x$. So what do we do? By induction, we want a common eigenvector for $\mathfrak{g}$. We can assume we already have such an eigenvector in $\mathfrak{h}$, namely $hv = \lambda(h)v, \lambda \in \mathfrak{h}^*$.

Let $W = \langle v, xv, \dots, x^K v \rangle$, so this generates all polynomials. If h and x commuted then this would be trivial. We don't know if this is an eigenspace. So $\mathfrak{h}$ preserves W and $hx^i v = \lambda(h)x^i v + \sum_{j<i} c_i x^j v$.

So $hxv = xhv + [h, x]v$, and so this is $(\lambda(h)xv + \lambda([h, x])v$. This was the first statement. You can easily get the rest yourself by induction.

The second thing is $\lambda([h, x]) = 0$. The trace satisfies $tr_W([h, x] = n\lambda([h, x]) = 0$.

The third thing is then that $hxv = \lambda(h)xv$ so that this whole space is an eigenspace for h.

So: $hw = \lambda(h)w$ for all $w \in W$. How do we find an eigenvector here for x? There's some eigenvector for x in W and then that does it. This is not an obvious proof. So what we had to do here was add these steps, and it's hard to explain why it works this way. Unfortunately I don't think there is a more intuitive proof of this fact.

Anyway, that's the end of the theorem. Whether we like it or not, there are many useful corollaries.

Corollary 1 *Any irreducible representation of a solvable Lie algebra is* 1*-dimensional.*

This is immediate; we have 1-dimensional subspaces invariant under the action. You can't say that this acts trivially, but it must be 1-dimensional. This is only for irreducible representations. The representations, generally, don't split into irreducibles.

What else? Apply this to the *ad* representation.

Corollary 2 *If* $\mathfrak{g}$ *is solvable then there exists a sequence* $0 \subset I_1 \subset \cdots \subset I_n = \mathfrak{g}$ *with* I_k *an ideal in* $\mathfrak{g}$ *and* I_{k+1}/I_k *is one dimensional. This is somewhat stronger than our definition.*

The third corollary, which we get almost for free, is the following.

Corollary 3 *If* $\mathfrak{g}$ *is solvable, then* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Again consider the adjoint action of $\mathfrak{g}$ on itself. Then there is a basis wherein $ad\ x$ is upper triangular. Then $ad\ [x, y] = [ad\ x, ad\ y]$ is the commutator of two upper triangular matrices, so is strictly upper triangular. Then since we know that the algebra of strictly upper triangular matrices is nilpotent, we are almost done. Why do I say almost? We haven't shown that the commutator is 0. So $[y, [x_1, \dots, x_n] \dots] = 0$ for all $y \in \mathfrak{g}$.

I think that's all we need to know about nilpotent and solvable algebras. Their irreducibles are pretty obvious, but if we want to describe all the representations, not just the reducibles, then we run into problems. We don't have complete reducibility, but we do have a nice set of irreducible ones.