Introduction to Lie Groups and Lie Algebras
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Recall: $\mathfrak{g}$ is a semisimple complex Lie algebra, and $\mathfrak{h} \in \mathfrak{g}$ is a Cartan subalgebra so that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Then $R \in \mathfrak{h}^*$ has a number of properties:

Definition 1 *Let E be a finite dimensional real Euclidean space. A finite subset $R \subset E \backslash \{0\}$ is called a root system if it satisfies*

1. *R spans E.*
2. *If $\alpha, \beta \in R$ then $S_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \langle \alpha^\vee, \beta \rangle \alpha \in R$. This is reflection in the hyperplane perpendicular to α. Here, also, $\alpha^\vee \in E^*$ is defined by the above by $\langle \alpha^\vee, \lambda \rangle = \frac{2(\alpha,\lambda)}{(\lambda,\lambda)}$.*
3. *If $\alpha, \beta \in R$ then $\langle \alpha^\vee, \beta \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.*

R is called reduced if $\alpha \in R$ implies $2\alpha \notin R$.

Some remarks:

1. If $R_1 \subset E_1, R_2 \subset E_2$ are root systems then $R_1 \sqcup R_2 \subset E_1 \oplus E_2$ is a root system.
2. If we rescale a root system $\alpha \mapsto c\alpha$ for $c \in \mathbb{R}_*$ it will again be a root system.
3. The third part of the definition means that the projection of β on α is an integer multiple of $\frac{\alpha}{2}$.

Example 1 *(Pictures)*

Assume we are working with reduced root systems. Back to Lie algebras, let $\mathfrak{g}$ be as above. So $\mathfrak{h}^* = \mathfrak{h}^*_{\mathbb{R}} \oplus i\mathfrak{h}^*_{\mathbb{R}}$. Then $(\ ,\ )|_{\mathfrak{h}^*_{\mathbb{R}}}$ is positive definite and $R \subset \mathfrak{h}^*_{\mathbb{R}}$ is a reduced root system.

Example 2 *Take $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. Then take $\mathfrak{h}$ to be traceless diagonal matrices*

$$\left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix} \mid \sum \lambda_i = 0 \right\} \subset \mathbb{C}^{n+1}.$$

We'll index the roots by $\alpha_{ij}, i \neq j$. Let $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$. Then $[h, E_{ij}] = (\lambda_i - \lambda_j)E_{ij} = (\epsilon_i - \epsilon_j)(h)E_{ij}$, where $\epsilon_i(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix}) = \lambda_i$.

Then $\mathfrak{h}^* = \mathbb{C}\epsilon_1 \oplus \cdots \oplus \mathbb{C}\epsilon_{n+1} / \sum \epsilon_i = 0$.

So the root system is $\{\epsilon_i - \epsilon_j, i \neq j\}$.

$\mathfrak{h}_{\mathbb{R}} = \{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix} | \lambda_i \in \mathbb{R}, \sum \lambda_i = 0\}$.

So $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\epsilon_1 \oplus \cdots \oplus \mathbb{R}\epsilon_{n+1} / \sum \epsilon_i = 0$.

Theorem 1 *Let $\alpha, \beta \in R$ and $\alpha \neq c\beta$. Then, up to interchanging α, β, we must have one of the following.*

1. *$\alpha \perp \beta$ (like $A_1 \times A_1$).*
2. *(a) $|\alpha| = |\beta|, \phi = \pi/3$.*
   (b) $|\alpha| = |\beta|, \phi = 2\pi/3$.
   These cases correspond to A_2.
3. *(a) $|\alpha| = \sqrt{2}|\beta|, \phi = \pi/4$.*
   (b) $|\alpha| = \sqrt{2}|\beta|, \phi = 3\pi/4$.
   These cases correspond to B_2.
4. *(a) $|\alpha| = \sqrt{3}|\beta|, \phi = \pi/6$.*
   (b) $|\alpha| = \sqrt{3}|\beta|, \phi = 5\pi/6$.
   These cases correspond to G_2.

Look at $n_{\alpha\beta} = \langle b^\vee, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. This is $\frac{2|\alpha||\beta|}{|\alpha|^2} \cos\phi = \frac{2|\beta|}{|\alpha|} \cos\phi$. Then $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\phi$. The options are

- $n_{\alpha\beta} = n\beta\alpha = 0$. This is case one.
- $n_{\alpha\beta} = \pm 1, n\beta\alpha = \pm 1$ so that $\cos^2\phi = \frac{1}{4}$ and $\cos\phi = \pm\frac{1}{2}$. This is case two above.
- $n_{\alpha\beta} = \pm 1, n\beta\alpha = \pm 2$ so that $\cos^2\phi = \frac{1}{2}$ and $\cos\phi = \pm\frac{1}{\sqrt{2}}$. This is case three above.
- $n_{\alpha\beta} = \pm 1, n\beta\alpha = \pm 3$ so that $\cos^2\phi = \frac{3}{4}$ and $\cos\phi = \pm\frac{\sqrt{3}}{2}$. This is case four above.

Theorem 2 *The only rank two root systems are* $A_1 \times A_1, A_2, B_2, G_2$.

Let R be a rank two root system. Take $\alpha, \beta \in R$ such that the angle between them is the smallest possible. Then we must be in one of the cases $1, 2a, 3a, 4a$ (to eliminate the b cases reflect to get a root with smaller angle).

If we are in the case $2a$, by reflections it must contain all of the six roots in A_2; if it contained anything else we'd get a pair of roots with smaller angle. Similarly for $3a, 4a$.

Now let $t \in E$ such that $(t, \alpha) \neq 0$ for all $\alpha \in R$. Then $R = R_+ \sqcup R_-$, where $R_+ = \{\alpha \in R | (\alpha, t) > 0\}$ and $R_- = \{\alpha \in R | (\alpha, t) < 0\}$.

Example 3 *Look at the root system* A_n *of* $\mathfrak{sl}(n+1, \mathbb{C})$. *Take* $t \in \mathfrak{h}^*_{\mathbb{R}}$ *as* $t = (t_1, \cdots, t_{n+1})$ *with* $t_1 > \cdots > t_{n+1}$. *Then* $(t, \alpha_{ij}) = (t, \epsilon_i - \epsilon_j) = t_i - t_j$, *so this is positive when* $i < j$, *negative when* $i > j$.

Then $R_+ = \{\alpha_{ij} | i < j\}; R_- = \{\alpha_{ij} | i > j\}$. *So* $\oplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ *is the set of strictly upper triangular matrices, and* $\oplus_{\alpha \in R_-} \mathfrak{g}_\alpha$ *is the set of strictly lower triangular matrices.*

Definition 2 *A root* $\alpha \in R_+$ *is simple if it cannot be written as a sum of positive roots.*

Let $\Pi = \{\alpha_1, \cdots, \alpha_k\}$ be the set of simple roots.

Lemma 1 *Any positive root can be written as* $\alpha = \sum n_i \alpha_i$ *for* $n_i \in \mathbb{Z}_+$.

Lemma 2 $(\alpha_i, \alpha_j) \leq 0$.

Lemma 3 $\{\alpha_i, \ldots, \alpha_k\}$ *are linearly independent*

The proof is an exercise.

In $\mathfrak{sl}(n+1)$, the simple roots are $\epsilon_1 - \epsilon_2, \cdots, \epsilon_n - \epsilon_{n+1}$.