Introduction to Lie Groups and Lie Algebras
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By the way, there is a misprint in the homework assignment. The first part of problem 2b is incorrect as stated, but can be corrected. I had a professor who would give me the problem to show that two things are the same, and if I would come back and say that they were not the same, he would tell me to change the question so that they were the same. I was not often successful, but in this case it is easy.

Today is probably the most important lesson we've had so far.

Recall that $\mathfrak{g}$ is a semisimple complex Lie algebra. There exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_\alpha, \alpha \in \mathfrak{h}^*$ where $\mathfrak{g}_\alpha$ is defined so that $[h, x] = \alpha(h)x$ for $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha$.

So you can forget everything else, Killing forms and Cartan criterion and so on; this is the most important part of being a semisimple Lie algebra.

These α are called roots, and the root system R is $\{\alpha : \mathfrak{g}_\alpha \neq 0\} \subset \mathfrak{h}^* - \{0\}$. $\mathfrak{g}_\alpha$ is called the root subspace. I don't know where this terminology came from but it is absolutely standard and could not be changed at this point.

Example 1 *1.* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.
$\mathfrak{h} \cong \mathbb{C}h, \mathfrak{h}^* \cong \mathbb{C}$
$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{C}e \oplus \mathbb{C}f$. *The first of these corresponds to* $\alpha(h) = 2$, *the second to* $\alpha(h) = -2$.

Here every root space is one dimensional.

2. *As a slightly more complicated example let* $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. *Then I claim that* $\mathfrak{h}$ *consists of diagonal traceless elements.*

   You can write $\mathfrak{g}$ *as* $\mathfrak{h} \oplus \bigoplus_{i \neq j=1}^3 \mathbb{C}E_{ij}$. *I claim that this is a root decomposition. If you fix* h, *with entries* λ_i, *you can compute that* $[h, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$. *You can describe the corresponding roots. If you define* $e_i \in \mathfrak{h}^*$ *by* $e_i(h) = \lambda_i$, *then first these do not form a basis, since* $\mathfrak{h}^*$ *has* $e_1 + e_2 + e_3 = 0$. *They are still well-defined elements, and we can say that* $E_{ij} \in \mathfrak{g}_{e_i - e_j}$. *So there are six root spaces, each one dimensional. I can't draw the complex space but I can draw the real part. Let me consider the original* $\mathbb{R}^2$, *with three vectors dividing the plane at $2\pi/3$ angles. Call these e_1, e_2, e_3, and then they satisfy the relation. There are then six roots seperating by $\pi/3$ angles into sixths. This is known as A_2.*

This suggests that we're going to get nice and symmetrical pictures. Let's see what we can prove about roots and all that. Let me remind you of what we had last time.

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
2. $K : \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondegenerate. In particular, K is nondegenerate on $\mathfrak{h}$. We'll just use $(,)$ to denote it.

   I should note that a nondegenerate symmetric bilinear form on a vector space you can use it to identify the space with its dual via $\mathfrak{h}^* \leftarrow \mathfrak{h}$ by $\alpha \leftarrow H_\alpha$. Then you can define $(,)$ on $\mathfrak{h}^*$ so that $(\alpha, \beta) = (H_\alpha, H_\beta)$.

So what can we prove about the root system? Here is the first result.

Proposition 1 *R spans $\mathfrak{h}^*$.*

Suppose not. Then there is a nonzero $h \in \mathfrak{h}$ with $\langle h, \alpha \rangle = 0$ for all $\alpha \in R$. Then if I take $[h, x]$ it will be 0 for all $x \in \mathfrak{g}_\alpha$, since it is $\langle h, \alpha \rangle x$.

Then h is central, since it commutes with everything; but then h is not semisimple, which is impossible.

What's next? In both of the examples, we'd seen

Proposition 2 *For any $\alpha \in R$, if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. Then $[x, y] = (x, y)H_{alpha}$.*

This should remind you of the commutation relation $[e, f] = h$ in $\mathfrak{sl}(2)$. Well, we know $[x, y] \in \mathfrak{g}_0 = \mathfrak{h}$. To compute which it is, I need to look at $([x, y], h)$, which will work because the Killing form is nondegenerate. From invariance of the Killing form this is $-(y, [x, h])$. This is $(y, [h, x]) =< h, \alpha > (y, x) = (x, y)(h, H_\alpha)$. These have the same inner product with any element of Cartan so the Killing form is nondegenerate.

Proposition 3 *$(\alpha, \alpha) \neq 0$ for any α. Moreover, if we choose $e_\alpha \in \mathfrak{g}_\alpha, f\alpha \in \mathfrak{g}_{-\alpha}$ so that $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$, and let $h_\alpha = \frac{2H_\alpha}{(\alpha,\alpha)}$ then $(e_\alpha, f_\alpha, h_\alpha)$ satisfy the relations for $\mathfrak{sl}(2, \mathbb{C})$.*

It's nondegenerate but complex, so it might give you zero squares, but not roots. This is the most important proposition so far. So let's see. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, and $(x, y) = 1$. Then $[x, y] = H_\alpha$. We have $[H_\alpha, x] = \langle H_\alpha, \alpha \rangle x - (\alpha, \alpha)x$. On the other hand, $[H_\alpha, y] = -(\alpha, \alpha)y$.

If $(\alpha, \alpha) = 0$ then (x, y, H_α) is solvable.

Consider $\mathfrak{g}$ as an L-modules. In a suitable basis the *ad*s of these three elements are upper triangular. Then their commutator is strictly upper triangular, so that H_α is strictly upper triangular but diagonalizable.

On the other hand H_α is a semisimple so $H_\alpha = 0$, which is impossible because α was nonzero.

Now the relations of $\mathfrak{sl}(2,\mathbb{C})$ are rather immediate. Let me erase again. Then $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$ implies $[e_\alpha, f_\alpha] = \frac{2H_\alpha}{(\alpha,\alpha)} = h_\alpha$. Then $[h_\alpha, e_\alpha] = \langle h_\alpha, \alpha\rangle e_\alpha = \frac{2}{(\alpha,\alpha)}\langle H_\alpha, \alpha\rangle e_\alpha = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)} e_\alpha = 2e_\alpha$.

So for each root we have an $\mathfrak{sl}(2)$ algebra, which is extremely important. So you can consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2,\mathbb{C})$.

Proposition 4 *$\eta_{\beta\alpha} = \langle h_\alpha, \beta\rangle \in \mathbb{Z}, \alpha, \beta \in R$, which is the quantity $2(\alpha,\beta)/(\alpha,\alpha)$.*

So I claim these are integers. Consider $\mathfrak{g}$ as a module over $(e_\alpha, f_\alpha, h_\alpha$. These are $h_\alpha|_{\mathfrak{g}_p}$. This means that these inner products are real.

Proposition 5 *If $\alpha, \beta \in R$ then $s_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$. The constant is chosen so that $(s_\alpha(\beta), \alpha) = -(\beta, \alpha)$.*

Consider $(ad\ f_\alpha) : \oplus \mathfrak{g}_{\beta+k\alpha} \to \mathfrak{g}$. This is stable under the triple $e_\alpha, f_\alpha, h_\alpha$. Then $h_\alpha|_{\mathfrak{g}_{\beta+k\alpha}} = \langle h_\alpha, \beta\rangle + 2k$.

As part of your homework, we have a pairing of $\alpha, -\alpha$. Now the pairing will be of β and $s_\alpha(\beta)$.

Proposition 6 *Let $\alpha \in R$. Then $\dim \mathfrak{g}_\alpha = 1$ and $k\alpha \in R$ if and only if $k = \pm 1$.*

From the fourth proposition we get that $\frac{2(k\alpha,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, so that k must be a half-integer. I can also write $\alpha = \frac{1}{k}\beta$ to get $\frac{1}{k}$ also a half-integer. The only possibilities are $k = \pm 1, \pm 1/2, \pm 2$. Without loss of generality, we need only say $\alpha \in R, 2\alpha \in R$ as impossible. Look at $V = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Then V is a module over $(e_\alpha, f_\alpha, h_\alpha)$.

We can even compute the eigenvalues for h_α for each of these; they are $-4, -2, 0, 2, 4$. What can we say about this representation? Notice that $V[0]$ is one dimensional. This tells you that it's irreducible. This is homework. Since we know that these are nonzero, this is the four dimensional representation V_4 of $\mathfrak{sl}(2,\mathbb{C})$. So all these spaces are one-dimensional. But then $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$, and $ad\ e_\alpha : \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is a zero map. But in an irreducible representation, e_α cannot act by 0 so this cannot happen.

Let me write the last result, which I don't have time to prove. Maybe I'll prove it next time or maybe I'll leave it as a statement to be proved as an exercise.

Proposition 7 *Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-span of α, and $\mathfrak{h}_{\mathbb{R}}$ the $\mathbb{R}$-span of h_α.*
Then

1. $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}, \mathfrak{h}^* = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$.
2. $(,)|_{\mathfrak{h}_{\mathbb{R}}^*}$ *is positive definite.*

Next time we'll summarize and see what we can say about root systems from this.