Introduction to Lie Groups and Lie Algebras
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I will give you your last homework assignment on Thursday. That will be the last one.

[Is the Weyl group for D_n symmetric?]

No. Instead of thinking of it as generated by simple reflections, you think of it as generated by all reflections. The reflections around hyperplanes of form $w_i + w_j$ will be composition of transposition and a change of sign. Therefore your group will be a semidirect product of the permutations with an appropriate group of sign changes, in an even number of components. So it will be a semidirect product with Z_2^{n-1}. Other than that things look good.

As you know we didn't have a class last time, so we'll have to make up for it.

Since it has been a while let me remind you where we stopped. We started with a semisimple Lie algbera $\mathfrak{g}$ and we constructed a Cartan subalgebra and got the decomposition $\mathfrak{h} \oplus \mathfrak{g}_\alpha$, which gave us the root system $R \subset \mathfrak{h}^*_{\mathbb{R}}$, which was reduced, so twice a root was not a root, and this gave us a system Π of simple roots, which gives a Cartan matrix, or what is essentially the same thing, a Dynkin diagram, so that if you started with a simple Lie algebra you would get an irreducible root system which is the same as a connected Dynkin diagram. All of these are contained in the set $\{A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2\}$.

Since we know that semisimple Lie algebras are the direct sums of simple ones, we can just classify simple ones. We know how to get from a Dynkin diagram to a root system, by beginning with the simple roots and then using the Weyl group to generate all of the roots. But can you reconstruct the Lie algebra from the root system? If two Lie algebras have the same root system are they isomorphic and does every root system correspond to a Lie algebra? The answer is yes. Every root system arises as the root system of a semisimple Lie algebra. Once we know this we will have classified all semisimple Lie algebras.

For today let $\mathfrak{g}$ be a simple Lie algebra; most of what I say will work for semisimple Lie algebras with obvious changes. Let $g = \mathfrak{h} \oplus \oplus_\alpha \mathfrak{g}_\alpha,\ R \subset \mathfrak{h}^*_R,\ R = R_+ \cup R_-, \Pi \subset R_+$, and we want to go backward.

Let me erase the general picture.

Lemma 1 *Let $n_\pm = \bigoplus_{\alpha\in R_\pm} \mathfrak{g}_\alpha \subset \mathfrak{g}$. Then $\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+$ (direct sum as vector spaces, and each is a subalgebra)*

This rather easy lemma says that if you seperate $\mathfrak{g}$ into the Cartan algebra, which we think of as diagonal, the positive, which we think of as upper triangular, and the negative, which we think of as lower triangular, then each is a subalgebra.

The proof is immediate. We know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. If both α and β are positive then so is $\alpha + \beta$, so that's it. The same argument of course works for negative roots. The Cartan subalgebra is Abelian so every commutator is zero.

That doesn't quite answer our question, to recover the Lie algebra from the root system.

Again, let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ be Cartan and all that. Choose e_i corresponding to each of the simple root spaces $\mathfrak{g}_{\alpha_i}$, and $f_i \in \mathfrak{g}_{-\alpha_i}$. This is unique up to a constant; we choose them so that $(e_i, f_i) = \frac{2}{(\alpha_i,\alpha_i)}$. This second is the Killing form on $\mathfrak{h}_*$. You still have choice, but not too much. Also let $h_i = \frac{2H_{\alpha_i}}{(\alpha_i,\alpha_i)} \in \mathfrak{h}$. This is from the identification of $H_\alpha \in \mathfrak{h}$ with $\alpha \in \mathfrak{h}^*$.

Theorem 1

1. *Elements e_i, f_i, h_i generate $\mathfrak{g}$. Moreover, e_i generate n_+, f_i generate n_-, and h_i generate $\mathfrak{h}$.*
2. *Of course we would also like to know the relations, and that is where the fun really begins. Let $a_{ij} = \langle h_i, \alpha_j\rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$. This is linear in α_j but not really in α_i, although that doesn't matter. Then the following relations are satisfied in $\mathfrak{g}$:*
   (a) *$[h_i, e_j] = a_{ij}e_j$. We know it is e_j but what is the eigenvalue? It is the value of h_i on the corresponding root. Also $[h_i, f_j] = -a_i f_i$, and $[h_i, h_j] = 0$. This is the easy part.*
   (b) *$[e_i, f_j] = \delta_{ij}h_i$. If $i = j$ then these are something we have discussed before. Every root defines an $\mathfrak{sl}(2)$ triple. In $\mathfrak{sl}(2)$ we have $[e, f] = h$ so this is the proof we had before. You need to go back to that proof and check that we have the same constants as before. Now why is it zero if I take e_i with f_j if they are not equal. If $i \neq j$ then $[e_i, f_j] \subset \mathfrak{g}_{\alpha_i-\alpha_j}$, but this is not a root. Every root is written as a sum of positive simples or negative simples, but not mixed ones.*
   (c) *$(ad\ e_i)^{1-a_{ij}}e_j = 0$ and $(ad\ f_i)^{1-a_{ij}}f_j = 0$.*

How do you show that these generate the Lie algebra. We already know, choose $\alpha, \beta \in R$ distinct so that we have $sl(2) \hookrightarrow \mathfrak{g}$ and each root defines a subalgebra defined by $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ and the appropriate element of the Cartan subalgebra.

If I now take the direct sum $\oplus_{k\in\mathbb{Z}}\mathfrak{g}_{\beta+k\alpha}$, if I take this thing I get a subspace, not an algebra, but it is a module over the triple. It is more than a module; it is an irreducible $\mathfrak{sl}(2)_\alpha$-module.

We already know that each of these spaces is one dimensional. A module over $\mathfrak{sl}(2)$ where every weight space is one dimensional is irreducible. I'm cheating a little bit, but you basically had this as homework. So what?

In particular, it implies the following thing that should have been proven before. If $\alpha+\beta \in R$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is either $\mathfrak{g}_{\alpha+\beta}$ or is 0. Why is it not zero? We know everything about representation theory of $\mathfrak{sl}(2)$. We know that the action of e is nonzero except to the highest weight vector. So the action is nonzero. So in an $\mathfrak{sl}(2)$ module all of the arrows between weights are nonzero. Why am I doing this? It immediately tells me why the e_i generate n_+.

If $\alpha \in R_+$ then $\mathfrak{g}_\alpha$ can be obtained by taking commutators of simple roots, that is, of e_i. That will prove part one.

Why? By induction in length of α as a sum of simple roots. If the root is simple there is nothing to prove. If it is not, according to the homework it can be written $\alpha_i+\beta$ where $\beta \in R_+$ and α_i simple. Therefore by this argument, $\mathfrak{g}_\alpha = [e_i, \mathfrak{g}_\beta]$. By the induction assumption this can be obtained by commutators of the e_i. That proves the theorem part one.

It is the same, obviously for f_i.

For part 2c, let me give an example. Say we have a root system of type A_n. Then a_{ij} are -1 and $i=j=1$. If you take two roots which are far from each other you get $a_{ij}=0$. Next to each other they are -1. The relations are, if $a_{ij}=0$ then $[e_i,e_j]=0$, so that e_i and e_j commute. The relation otherwise is $[e_i,[e_i,e_j]]=0$.

You can check that this is so because we know the root vectors in $\mathfrak{sl}(n)$. The e_i will be a 1 on the i superdiagonal. It is not that hard to get these relations.

How do you prove this in general? We need to prove all of these relations, let's talk about the only one of any interest. By the way, these relations are commonly called Serre relations. What we need to do is the following. Consider $\bigoplus_k \mathfrak{g}_{\alpha_j+k\alpha_i}$. This will be an irreducible module over $\mathfrak{sl}(2)_{\alpha_i}$. So what is the highest and lowest weight? Every $\mathfrak{sl}(2)$ module should consist of weights $n, n-2, \cdots, -n$. I claim that $\mathfrak{g}_{\alpha_j}$ is the lowest you can get. What would the previous one be? It would be $\alpha_j-\alpha_i$ which is not a root because the difference of simple roots is not a root. So $\mathfrak{g}_{\alpha_j}$ is the lowest weight. So $V=V_{n_j}$. To get the weight you should take the eigenvalue of the lowest one with a minus sign. Then what is $-h_i\alpha_j$? It is $-a_{ij}$.

Forget this picture; if I have an irreducible $\mathfrak{sl}(2)$ module, how many times to apply it to get to the highest weight? n times. Then $e^{n+1}=0$. So if I have a vector of weight n and I apply e to the power of the lowest weight plus one, I get zero. Therefore $(ad\ e_i)^{n+1}\mathfrak{g}_{\alpha_j}=0$. Then the Serre relation comes from combining two trivial steps. That's the end of the story for Serre relations.

The natural question is whether there are more relations. Let me just quote a result:

Theorem 2 *(Serre)*

1. *Let R be a reduced root system and $\mathfrak{g}(R)$ a complex Lie algebra generated by e_i, f_i, h_i with the relations above. Then $\mathfrak{g}$ is finite dimensional semisimple with root system R.*
2. *if $\mathfrak{g}$ is a semisimple Lie algebra then $\mathfrak{g}$ is generated by e_i, f_i, h_i with the above relations.*

So I don't know which order to put them. First of all this is a full set of relations. A Lie algebra freely generated subject to these relations, which you can get by the quotient of a free Lie algebra will be your Lie algebra $\mathfrak{g}$. The Lie algebra is generated by them and this is a full set of relations. If you have a Lie algebra generated by e_i, f_i, h_i with these relations, it will be a semisimple finite dimensional Lie algebra.

I don't think I can prove this part. The hard part is to show that it is finite dimensional. In particular, if I forgot to impose the Serre relations it would not be finite dimensional. The easiest way to talk about it will be with representations of finite weight. Once you know the first part, the second part is easy to show.

Okay, like I said I'm not proving that, you can find the proof in many places. If you want, look it up in Serre's book, Humphrey's book, Jacobson's book.

So what we have now is that from a root system you can recover your Lie algebra and every root system defines a Lie algebra.

Corollary 1 *For each irreducible root system R there is a unique (up to isomorphism) simple Lie algebra $\mathfrak{g}$ with root system R.*

Can we explicitly describe the Lie algebras? In principle we can go back and get our Lie algebras? Is there such a thing? You can write the corresponding Lie algebras.

- A_n has $\mathfrak{sl}(n+1, \mathbb{C})$.
- B_n has $\mathfrak{so}(2n+1, \mathbb{C})$
- C_n has $\mathfrak{sp}(2n, \mathbb{C})$
- D_n has $\mathfrak{so}(2n, \mathbb{C})$

The others can be constructed explicitly as subalgebras of $\mathfrak{sl}$ of a certain rank, but they are not very illuminating. The common way is by referring to these symbols. In physics they will talk about the Lie algebra E_8 or G_2, by which they mean a Lie algebra with this root system. If you want to see the description, by all means there are places to find them.

So let me say a little bit about how these four classical systems are obtained. Let me talk about $\mathfrak{sl}$, well, we already know. Let me talk about $\mathfrak{so}$. In these cases it easiest to write your Lie algebra not as skew-symmetric matrices but rather write $\mathfrak{so}(n, \mathbb{C})$ (any two symmetric nondegenerate bilinear forms can be obtained one from another by a change of variables) as $B(x,y) = \sum x_i y_{i+n} + x_{i+n} y_i$. So you take the bilinear form with the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. For odd you take the matrix $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then the corresponding matrices will be $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ with the B, C skew symmetric.

The Cartan subalgebra is diagonal matrices, and in the odd case they will have zero in the bottom corner.

How many of you did the homework? The roots are as close to matrix units as you can get. The root subspaces will be pairs of $1, -1$ in transposed positions in B or C or 1 somewhere off the diagonal of A and in the corresponding position in the lower right matrix.

So these are the matrix units for $\mathfrak{so}$. Again if you did the homework the corresponding roots are $\pm\epsilon_i \pm \epsilon_j$, and this will give D_n. If you have odd coefficient, you will have all the same things but in addition, let me erase this, for $\mathfrak{so}(2n+1, \mathbb{C}) = \left\{ \begin{pmatrix} A & B & \vec{x} \\ C & -A^t & \vec{y} \\ -\vec{x}^t & -\vec{y}^t & 0 \end{pmatrix} \right\}$.

If you look what are the roots corresponding to this one you get $\pm i \pm j$ for $i \neq j$ and also $\pm e_i$ for the last row or column. You have to work it out, it is not simply laced; then you get a root system of type B_n. The most difficult part was how to choose the Cartan.

For C_n it is very similar to what you have here except you have $-I$ so you get symmetric matrices as B, C and so on. But the Cartan will basically be the same thing.

It is not very difficult in each of these cases but I don't want to go over them myself here. Any of the books I quoted give you all of the details of this argument. It is all written in countless books and you can find the same for the exceptional Lie algebras. It's all nicely classified. After all the classification is given by a nice combinatorial picture.

I think that completes what I wanted to say about classification of semisimple Lie algebras. What we'll do in the remainder of the semester is representation theory of simple Lie algebras. We know $\mathfrak{sl}(2)$ which is nice but not exactly what I want.

I will assign new homework on Thursday.