Introduction to Lie Groups and Lie Algebras
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I gave a midterm in the other class and of course afterward I had to discuss all the grades they got. Here is the next homework; much of it has to do with the Laplace operator on the sphere. I discussed this the first day and it took us two months to get there.

Recall that we were discussing representations of $\mathfrak{sl}(2,\mathbb{C})$. These representations are indexed by nonnegative integers $k \in \mathbb{Z}_+$ where V_k is a representation of dimension $k+1$. Then every representation is the direct sum of these with multiplicity. We found these by looking at how e and f shift the eigenvalues of h. The key was to have an analogue of the relations defining $[h,e]$ and $[h,f]$. The key was to have a semisimple element.

Definition 1 *$x \in \mathfrak{g}$ is semisimple if $ad\ x$ is a semisimple (diagonalizable) operator $\mathfrak{g} \to \mathfrak{g}$ so that there is a basis $\{y_i\}$ with $ad\ x\ y_i = \lambda_i y_i$.*

Definition 2 *$x \in \mathfrak{g}$ is called nilpotent if $ad\ x$ is nilpotent.*

Exercise 1 *These definitions coincide with the usual ones for $g = \mathfrak{g}$.*

Theorem 1 *If $\mathfrak{g}$ is semisimple then any $x \in \mathfrak{g}$ can be uniquely written in the form $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent.*

Moreover, $ad\ x_s = p(ad\ x)$ for some $p \in t\mathbb{C}[t]$.

I'll give a sketch of the proof. Consider $ad\, x_s : \mathfrak{g} \to \mathfrak{g}$. Let $\mathfrak{g} = \oplus \mathfrak{g}_{(\lambda)}$, with $(ad\, x - \lambda)^n|_{\mathfrak{g}_{(\lambda)}} = 0$. When you have a Lie algebra you can get more. I claim that $[\mathfrak{g}_{(\lambda)}, \mathfrak{g}_{(\mu)}] \subset \mathfrak{g}_{(\lambda+\mu)}$.

Did I do this argument last time where x was semisimple? Okay, so if $ad\ x\ y = \lambda y, ad\ x\ z = \mu z$, then $ad\ x\ [y,z] = [ad\ x\ y, z] + [y, ad\ x\ z] = (\lambda+\mu)[y,z]$.

Unfortunately I do have to consider generalized eigenspaces; there I have to be more careful. Here I would be able to say $(ad\ x - \lambda - \mu)^n [y,z] = \sum [(ad\ x - \lambda)^k y, (ad\ x - \mu)^{n-k} z]$. This is just the result of repeating the Liebnitz rule many times. This is just the nth derivative of a product, which is a binomial expansion. The meaning is slightly different, but the method is the same. By choosing high enough n you can make every term vanish.

I don't really feel like proving this formula. You can do it yourself.

Definition 3 *Let $a : \mathfrak{g} \to \mathfrak{g}$ be $a = (ad\ x)_s : a|_{\mathfrak{g}_{(\lambda)}} = \lambda id$.*

Then a is a derivation of $\mathfrak{g}$ so $a = ad\ y$ for some y.

Indeed, if $y \in \mathfrak{g}_{(\lambda)}, z \in \mathfrak{g}_{(\mu)}$, then

$$a[y, z] \stackrel{?}{=} [ay, z] - [y, az].$$

So the way I defined these we have $(\lambda + \mu)[y, z] = \lambda[y, z] + \mu[y, z]$.

That shows you that you can write $(ad\ x)_s$ as ad of some operator. I don't want to spend too much more time on this. This last step to go from being a derivation to being ad of something, is where you use semisimplicity.

Let me write a seperate corollary.

Corollary 1 *In any semisimple Lie algebra, there are nonzero semisimple elements.*

If $x_s = 0$ for all $x \in \mathfrak{g}$ then any $x \in \mathfrak{g}$ is nilpotent so by the Engel theorem, $\mathfrak{g}$ is nilpotent. Therefore if $\mathfrak{g}$ is semisimple then there exists such an x.

In $\mathfrak{sl}(n)$ we can just take diagonal elements. In general this shows you that there are semisimple elements. But now we need sufficiently many to be useful. Now that I have them, let me repeat what I tried for other Lie algebras. I started by choosing a basis to get h to be in the appropriate form.

Definition 4 *A toroidal subalgebra is a commutative subalgebra in $\mathfrak{g}$ which consists of semisimple elements.*

This language comes from Lie groups, where these are algebrae of torii.

Example 1 *Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Then Let $\mathfrak{h}$ be the diagonal matrices.*

We know these exist by the last theorem, but we can do better. What are the properties that we have?

Theorem 2 *Let $\mathfrak{h} \subset \mathfrak{g}_{ss}$ be toroidal. Then*

1. $\mathfrak{g} = \oplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda ad\ h\ x = \lambda h x, h \in \mathfrak{h}, x \in \mathfrak{g}_x$.

2. *In particular, $\mathfrak{g}_0$ is the cenetralizer of $\mathfrak{h}$ so contains $\mathfrak{h}$ itself.*

3. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu}$.

Let K be the Killing form. Then $K(x, y) = 0$ if $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu, \lambda + \mu \neq 0$.

$K : \mathfrak{g}_\lambda \otimes \mathfrak{g}_{-\lambda} \to \mathbb{C}$ is a nonempty pairing.

The first thing here is about diagonalizing them simultaneously, which uses basic linear algebra. Just remember that λ is no longer a number, but a vector in $\mathfrak{h}^*$.

The second is obvious; the third we proved more generally already. So what about the more recent? We have $K(x, y) = tr(ad\ x\ ad\ y)$. But such a product takes $\mathfrak{g}_. \to \mathfrak{g}_. + \lambda + \mu$.

That's it. For the fifth, it follows from the previous one and the nondegeneracy of the Killing form.

Finally we are essentially ready to define the notion of a Cartan subalgebra.

Definition 5 *A Cartan subalgebra of $\mathfrak{g}$ is a toroiral subalgebra $\mathfrak{h}$ such that it is precisely its own centralier.*

There is a different definition which specializes to this one for general algebras, not just semisimple ones.

Of course, it's not trivial that such algebras exist.

Theorem 3 *Carton subalgebras exist*

Theorem 4 *Any two such are conjugate.*

This second theorem really requires either complex algebraic geometry or pure analysis. There is no purely algebraic proof which does not require geometry or topology. Instead let me tell you why these things exist. If you want the proof you can find it anywhere.

Let's take $\mathfrak{h}$ to be a maximal toroidal subalgebra, maximal in the sense that it can't be included in a larger one. Then we know that the lie algebra can be written as $\oplus \mathfrak{g}_{(\lambda)}$, or

$$\mathfrak{g}_0 \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_{(\lambda)}.$$

Then I claim that $\mathfrak{g}_0$ is $\mathfrak{h}$ so that $\mathfrak{h}$ is a Cartan subalgebra. How do we prove it?

Step one is to show that $\mathfrak{g}_0$ is reductive, meaning that you are the direct sum of semisimple and commutative Lie algebras.

Let me cheat a little bit. Because $\mathfrak{K}|_{\mathfrak{g}_0\times\mathfrak{g}_0}$ is nondegenerate. Why is this cheating? Is this the Killing form of this acting on $\mathfrak{g}$ or only on $\mathfrak{g}_0 \times \mathfrak{g}_{(0)}$?

I'm trying to write everything up so you can check what I have written for the course on the web.

Anyway, step two is that $\mathfrak{g}_0$ is commutative; otherwise it is the direct sum of a commutative piece and a semisimple piece. Now $\mathfrak{h}$ commutes with all of $\mathfrak{g}_0$ so that $\mathfrak{h}$ is in the center. Then there is a step to show that if there is anything else of a certain form it contradicts maximality.

What is step three? Why can't $\mathfrak{h}$ be properly contained in $\mathfrak{g}_0$?

[Is $C(\mathfrak{h})$ a toroidal subalgebra?]

We don't know that every element there is semisimple yet.

We know that $\mathfrak{g}_0$ is commutative, $\mathfrak{h} \subset \mathfrak{g}_0$. Now I claim that $\mathfrak{g}_0$ consists only of semisimple elements. Indeed, everything is a sum of a semisimple and a nilpotent so it suffices to explain why there are no nilpotents here. If $x \in \mathfrak{g}_0$ is nilpotent, then $tr(ad\ x\ ad\ y) = 0$ for any $y \in \mathfrak{g}$. The product of two commuting operators, one of which is nilpotent, is itself nilpotent and thus has trace zero. This contradicts the nondegeneracy of the Killing form.

Now we are done. What have we proved? We've shown that $\mathfrak{g}_0$ is toroidal and contains $\mathfrak{h}$, so by maximality they coincide. I looked for other proofs, but this was the easiest one. It is still not particularly nice.

In most cases, though, we can explicitly produce Cartan subalgebras basically by hand.

Example 2 *Let* $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. *Then* $\mathfrak{h} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ *with* $\sum \lambda_i = 0$ *is a Cartan subalgebra.*

For a proof, if A commutes with a diagonal operator with distinct eigenvalues, then A is diagonal. Alternatively they are simultaneously diagonalizable.

Okay, so that's it, to find something that commutes with all diagonals it must be diagonal itself.

For $\mathfrak{su}(n)$ you get the same thing as before. What about $\mathfrak{so}(4, \mathbb{C})$, skew symmetric matrices. If this is too hard, start with $\mathfrak{so}(2)$. This is commutative so it's the whole thing. If you use the definition of semisimple in terms of $ad\ x$ then this is trivial.

One candidate would be to have blocks of $\mathfrak{so}(2)$. You have to show that it's semisimple as a 4×4 matrix. The one with blocks is diagonalizable easily, though. What is hard to see that this is maximal, i.e., that there is no larger one.

Next time we'll talk about what you can say when $\mathfrak{h}$ is a Cartan subalgebra. We'll also discuss the collection of eigenvalues. For now we'll stop here.