Introduction to Lie Groups and Lie Algebras
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Okay, so let me as usual recall what we are doing. We have discussed the structure theory of semisimple Lie algebras, and that gave us a root system. Once you have that you can divide into positive and negative $R_+ \sqcup R_-$, and then we get a subset $\Pi \subset R_+$ called the simple roots, from which we can recover all of the roots by means of the Weyl group.

Our goal today is to classify all possible root systems, which will essentially solve the problem of classification of all semisimple Lie algebras. So classifying root systems is equivalent to classifying possible sets of simple roots. Before I continue let me just say that you can always take orthogonal unions of existent simple root systems. That is, if R_1, R_2 are root systems then $R_1 \cup R_2 \subset E_1 \oplus E_2$ is a root system. So you always can construct new ones by means of orthogonal unions. Such root systems are called reducible and we will be discussing irreducible root systems.

I claim that R is reducible if and only if $\Pi = \Pi_1 \cup \Pi_2, \Pi_1 \perp \Pi_2$.

If R is reducible, this direction is obvious, since the whole root systems are orthogonal. They will be nonempty because then one of the R_i would be nonempty since that's where everything generated by reflection would live.

An element of the Weyl group will preserve E_1, E_2 so this is as claimed.

The proof in the other direction is that, if you assume the hypotheses, then let E_1 be the span of Π_1, E_2 the span of Π_2. So this is a direct sum decomposition. Let $R = R \cap E_1 \cup R \cap E_2$. Show that this works as an exercise.

Definition 1 *Let Π be a set of simple roots. The Cartan matrix is defined by $c_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle = \frac{2(\alpha_i,\alpha_j)}{\alpha_i,\alpha_i} \in \mathbb{Z}$.*

Then $c_{ii} = 2, c_{ij} \leq 0$ otherwise. For example for A_2 we get $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. In general, for $A_n = \{\epsilon_i - \epsilon_j : i \neq j\}$ we get a matrix with 2 on the diagonal and -1 on the subdiagonal and superdiagonal.

There is an even better way of carrying this information, which is a graph. We have a vertex for every simple roots. We will come up with edges according to the angles between them. The only possibilities with an obtuse angle are $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$. We join α, β by $0, 1, 2$, and 3 edges according to which of these cases we are in, and we give orientations to the double and triple edes by going from the longer to shorter sides.

This kind of diagram is called a Dynkin diagram. As we discussed, any other choice of simple groups is related by the action of the Weyl group, so that the Dynkin diagram is determined by the choice of the root system.

For A_n the Dynkin diagram is a string; for B_2 it is two nodes joined by a double edge. The matrix for B_2 is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.

Claim 1 1. *One can recover the Cartan matrix from the Dynkin diagram and vice versa.*

2. *The root system is irreducible if and only if the Dynkin diagram is connected.*

3. *if you are talking about irreducible root systems, the Dynkin diagram determines R uniquely up to isomorphism*

The second of these is obvious, since you don't connect orthogonal pieces, and disconnected diagrams give you a union of orthogonal subsets of the root set.

So first we want to recover the simple roots up to isomorphism, i.e., an orthogonal transformation. That requires inner products. The diagram almost tells them. It tells you the relative length of the roots. Since we are talking about irreducible root systems we know the length of all the roots, relatively.

The Dynkin diagram determines $|\alpha_i|^2, (\alpha, \alpha)$ up to an overall constant factor, so all of the simple roots.

Our basic sequence was to start from a root system to get simple roots to get Dynkin diagrams. The results of the last week or two give us the other directions. So classifying root systems is the same as classifying Dynkin diagrams.

The famous answer is that there is a very easy classification.

Theorem 1 1. *For an irreducible root system R its Dynkin diagram must be one of the following:*

- *A_n is single edges.*
- *B_n has a double edge pointing out at the end of a string.*
- *C_n has a double edge pointing in at the end of a string.*
- *D_n has a single-edged fork at the end of a string.*
  Those form a series. The exceptionals are:

- E_6
- E_7
- E_8
- F_4
- G_2

2. *Each of these does occur as a Dynkin diagram of a root system.*

I'm not going to give you the full proof, but just a part of it. I'll classify those which are simply-laced, i.e., with only single edges.

The basic idea begins with the idea that all roots have the same length. Assume it is $\sqrt{2}$. Then $C_{ij} = (\alpha_i, \alpha_j)$ so that it is positive definite. Now any submatrix is also positive definite. What it means geometrically is that if we take a subdiagram, a subset of vertices and all edges connecting them, that the corresponding Cartan matrix is positive definite.

So now we can start. We begin by showing that Γ contains no loops. If it did, you would make your matrix, or part of it, singular. An n-loop is called $\hat{A}_4$.

Next, you have no vertex of valence 4 or higher. If you have that then you get again a singular submatrix, in this case $\hat{D}_4$.

The next step is that it is impossible to have two vertices of valence three. If you did, you could collapse along the line connecting them to get a case similar to the previous one. If they are connected along $\beta_1, \ldots, \beta_n$, then $\beta = \sum \beta_i$ has $|\beta|^2 = \sum 2n - 2(n-1) = 2$. Then $(\beta, \alpha_i) = -1$. This yields a linear combination that gives zero.

So if it has no vertices of valence three, then it is of type A_n. If it has such a vertex then it must have three tails of length p, q, r. Another argument will show the following famous inequality: $1/p + 1/q + 1/r > 1$. So they cannot all be long. After that, some simple number theory gives $(1, 1, n), (1, 2, 2), (1, 2, 3), (1, 2, 4)$.

If the diagram is not simply laced, the basic idea is the same. You need to show that if there is a double edge there can be only one, and so on. So you can find a full argument in many books, but I don't think we need a full discussion.

How do I know that every of these cases really does appear. The only way to do that is to construct a root system for each. Let me describe the root systems of types A, B, C.

Explicit description of root systems of type A, B, C.

- $A_n : R = \{\epsilon_i - \epsilon_j : i \neq j\} \subset \mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1)$. So
  $\Pi = \{\epsilon_1 - \epsilon_2, \cdots, \epsilon_n - \epsilon_{n+1};$
  $W = S_{n+1}$.
- $B_n : R = \{\pm\epsilon_i \pm \epsilon_j, i \neq j, \pm\epsilon_i\}$. The fact that this is a root system will follow from the Weyl group $W = \langle \sigma_i, \tau_{ij} \rangle = S_n \ltimes \mathbb{Z}_2^n$.

Positive roots can be $\epsilon_i, \epsilon_i + \epsilon_j, (\epsilon_i - \epsilon_j, i < j) = \{\alpha : (\alpha, (n, n-1, \ldots, 1)) > 0\}$.

$\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n$.

For the type C root system it's almost exactly the same, but you include $2\epsilon_i$ instead of ϵ_i.

Next time I will go back to Lie algebras and say what you get there.