Introduction to Lie Groups and Lie Algebras November 16, 2004

Gabriel C. Drummond-Cole

November 30, 2004

First of all, let me answer the question for why we are so interested in eigenvalues for the spherical Laplace operator. Consider an atom. You have some positive charge here and you have the electrons going around in circles, and the actual quantum mechanical discussion is the space of functions on $\mathbb{R}^3$, this being the space of states, if you have just one electron, I'm simplifying a little, and the energy operator, the Hamiltonian H will be $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, which will be $\Delta_{sph}+$ a radial part. Here we have electric charges so this isn't just the Laplacian, but the other part from the charge only depends on r. We want to find the eigenvalues of this because in quantum mechanics, the observables such as energy and momentum have possible values equal to the eigenvalues of this operator. So you find eigenvalues by seperating variables, and then you need to solve two seperate problems, a radial part and a part of finding eigenvalues on the sphere. The spherical eigenvalues are not exactly right because we didn't talk about the radial part. But we can see that there is one possible state with the lowest energy level, and so on. This is a little bit of simplification, but the actual answer is twice as large.

This exactly explains why the periodic table, why the structure changes when you fill all the possible energy levels.

The fact that these are indexed by positive integers gives you a lot of information about the structure of the hydrogen atom, and if we did the full analysis for the atom including the radial part you would get an explanation of the line spectra you see and so on. This was due to Bohr and it was the first test of quantum mechanics. That all goes to the physics. As far as mathematicians are concerned, the eigenfunctions are unimportant but the values are quite important because they predict possible states for a physical system. I should probably give you a reference for where this problem is analyzed completely; I'll have such a reference for you next time.

Okay, guys, the next homework assignment is due next week because we have Thanksgiving afterward. I would ask you to turn it in next Tuesday.

Last time the general setup is that we are talking about the root system. We have a finite set in Euclidean space satisfying certain axioms; the motivation comes from Lie algebras but we can forget that. We split R into $R_+ \cup R_-$ and then choose a basis $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R_+$ where r is called the rank. Of course this is not unique, as positive just came from taking an inner product with an arbitrary vector, but the set of all decompositions or polarizations is in bijection with what we call Weyl chambers. This is just a connected component of the complement of the hyperplanes $H_\alpha = \{\lambda | (\lambda, \alpha) = 0\}$

For example, in A_2 there are six Weyl chambers. The correspondence is that given a Weyl chamber, the choice of vector determines the same set of positive roots, and you can also go backward, to find that $C_+ = \{\lambda : (\lambda, \alpha) > 0 \text{ for all } \alpha \in R_+\}$; the more exciting result is that this is equal to $\{\lambda : (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Pi\}$.

Another important notion was the Weyl group $W = \langle S_\alpha \rangle$, the reflection group.

Theorem 1 *1. W acts transitively on the set of Weyl chambers.*

2. $W = \langle S_i \rangle$ where $S_i = S_{\alpha_i}$.

3. $W\Pi = R$.

We discussed an example; when $R = A_n$, which corresponds to $\mathfrak{sl}(n)$ then $W = S_{n+1}$. Did we discuss the simple roots in this case? Recall that these are indexed by α_{ij}. Then the reflection corresponds to the transposition of the i, j entries.

What are the simple reflections? The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$, which is the cycle $(i(i+1))$. That's a pretty obvious statement in this case but this works in general. Then this part tells you that any root can be obtained from the simple ones by permutation.

Let me remind for you one of the important tools used in the proof. We tried to connect C_+ to some other Weyl chamber. So we proved that we can always connect to Weyl chambers by such a chain $C_n = S_{\beta_n} \cdots S_{\beta_1}(C_+)$. But we said since β_1 is a wall of a Weyl chamber it corresponds to a simple reflection. Since β_2 is a wall of C_1, which is obtained from C_+ by reflection. So β_2 is a reflection $S_{\beta_1}(\alpha_{i_2})$ os that this is $S_{\beta_1} S_{i_2} S_{\beta_1}$, so you can write this as a product of simple reflections $S_{i_1} \cdots S_{i_n}(C_+)$. Two different choices of Weyl chambers give you isomorphic structures, angles, lengths, inner product all the same.

So let me continue and do a couple more things.

So first of all let me say a little bit more about this picture. I would like to discuss how long such a chain should be, and that really requires the following definition.

Definition 1 *Suppose $w \in W$. Then $l(w)$ is the number of hyperplanes separating*

$$C_+, w(C_+) = |\{\alpha \in R_+ : w(\alpha) \in R_-\}|.$$

By definition this length is nonnegative and easily bounded above, which is one of your homework assignments.

What is the length of a simple reflection? If you reflect a hyperplane around a boundary component then they are separated only by that compoenent. In this case the only root which has different signs on $C_+, w(C_+)$ is $alpha_i$, since $S_i(\alpha_i)$ is negative and S_i permutes the other positive roots.

There are purely algebraic arguments which are less obvious here but maybe would be easier to write down.

Corollary 1 *Let* $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha \in E$. *Then* $(\alpha_i^v, \rho) = \frac{2(\rho,\alpha_i)}{(\alpha_i,\alpha_i)} = 1$.

The proof is easy. Apply to this element the reflection S_i. We get $S_i(\rho) = \rho - (\alpha_i^v, \rho)\alpha_i$. This gives us $s_i(\frac{1}{2}\sum_{\alpha\in R_+}\alpha) = \frac{1}{2}(-\alpha_i + \sum_{\alpha\in R_+\setminus\alpha_i}\alpha = \rho - \alpha_i$.

As an example, to compute this for $\mathfrak{sl}(n+1)$ we can say $\rho = (t_1, \ldots, t_{n+1})$ up to multiple. We can write the half sum of positive roots which gives me $\frac{1}{2}\sum_{i<j}\epsilon_i - \epsilon_j$ which is $\frac{n}{2}\epsilon_1 + \frac{n-2}{2}\epsilon_2 + \cdots - \frac{n}{2}\epsilon_n$.

So now if you prefer you can write it $(\frac{n}{2}, \frac{n-2}{2}, \cdots, -\frac{n}{2})$. I leave it to you to check that the inner product with any simple root is one, which is easy.

Theorem 2 $l(w)$ *is the length of the shortest expression of the form* $w = s_{i_1}\cdots s_{i_l}$.

The proof is based on two simple lemmas.

Lemma 1 *If* $w = s_{i_1}\cdots s_{i_l}$ *then* $l(w) \leq l$.

There are two ways of doing this; let me choose the geometric way. Each time we cross one wall in our movement from C_+ to C_l by crossing l walls. If you have such an expression you can connect C_+ to $W(C_+)$ with a sequence of adjacent chambers of length l. That actually proves it.

Lemma 2 *On the other hand going the other way must cross at least* l *hyperplanes*

That proves the inequality. You may cross a hyperplane twice without meaning to. If $l(s_{i_1}, \ldots, s_{i_l}) < l$ then we can write $-n/2e_{n+1}$ as $(n/2, \cdots, -n/2)$, an expression for w which is shorter than l.

Corollary 2 W *acts simply transitively on the set of Weyl chambers.*

This means that for any Weyl chamber the stabilizer is the trivial one. The proof is one line. Assume $w(C_+) = C_+$; then $l(w) = 0$ so that the shortest expression of w is as zero simple reflections, so it is the empty word.

Probably I should give you one example here. Let me draw only the Weyl chambers for A_2. I drew the root system rather small. The emphasis is on the hyperplanes and the Weyl chambers. How can I go from a Weyl chamber to its opposite? It takes three crossings and I can go in either direction. So there should be an element of the Weyl group of length three, but there should be two expressions for it in terms of simple generators.

In one direction you get $s_1(s_1(H_2)) = s_2(s_1s_2s_1)s_1 = s_2s_1s_2$. The other direction gives you $s_1s_2s_1$, so that by our theorem these are equal. That is in fact so. If you think of the the Weyl group as permutations in three letters, this is an identity, the Yang Baxter identity.

Let me finish with a couple of words about Lie group theory. I'll just state results. I was just talking about Weyl groups as generated by reflections.

Theorem 3 *1. Any element $w \in W$ is given by Ad^*g for some $g \in G$.*

2. Conversely if $g \in G$ such that $Ad^*g\mathfrak{h} = \mathfrak{h}$ then $Ad^*g|_{\mathfrak{h}^*} \in W$.

Recall the root system lives in $\mathfrak{h}^* \subset \mathfrak{g}^*$ not in $\mathfrak{g}$ so this is not Ad.

The same idea has another form; this statement is also quite useful, but the statement is quite complicated

Theorem 4 *Consider the symmetric algebra $S\mathfrak{g}$ which is polynomials on $\mathfrak{g}^*$ and we take the elements $(S\mathfrak{g})^G$ invariant under the adjoint action of G. Then this is in correspondence, and isomorphism, with $(S\mathfrak{h})^W$, polynomials on $\mathfrak{h}^*$ which are fixed under the action of the Weyl group.*

Suppose $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ then $\mathfrak{h}$ is the diagonal and the theorem says that conjugation invariant polynomial functions of a matrix and the right hand side are the S_n-invariant polynomial functions of the eigenvalues, the space of symmetric polynomials.

That's not immediately trivial even in this case. It's obvious you have an embedding, but equality is not clear. In general it's somewhat harder.