Introduction to Lie Groups and Lie Algebras November 11, 2004

Gabriel C. Drummond-Cole

December 3, 2004

Recall: $R = R_+ \cup R_- \subset E$ and $r = \dim E$. $\Pi = \{\alpha_1, \cdots, \alpha_r\} \subset R_+$ is the set of simple roots. For any $\alpha \in R_+,\ \alpha = \sum n_i \alpha_i$ for $n_i \geq 0$.

Example 1 $R = \{e_i - e_j, i \neq j\} \subset \mathbb{R}^{n+1}/(1, \cdots, 1)$.
$R_+ = \{e_i - e_j, i < j\}$.
$\Pi = \{\alpha_1 = e_1 - e_2, \cdots, \alpha_n = e_n - e_{n+1}\}$.

Can one recover R from Π, and for different choices of polarization do we get "different" Π?

Note that $R = R_+ \cup R_-$ depends only on the signs of (t, α). It changes when $(t, \alpha) = 0$, i.e., t crosses the hyperplane $H_\alpha = \{\lambda | (\alpha, \lambda) = 0\}$.

Definition 1 *A Weyl chamber is a connected component of* $E \backslash \cup H_\alpha$.

Each Weyl chamber is the intersection of half-spaces so it is a convex polygonal cone. There are only finitely many.

Lemma 1 $C \to R = R_+^C \cup R_-^C$, *where* $R_+^C = \{\alpha : \alpha(C) > 0\}$ *is a bijection between Weyl chambers and polarizations.*

The inverse map is $R = R_+ \cup R_-$ to $C_+ = \{t \in E | (\alpha, t) > 0$ for all $\alpha \in R_+\} = \{t \in E | (t, \alpha_i) > 0$ for all $\alpha_i \in \Pi\}$.

If $(t, \alpha_i) > 0$ for all $\alpha_i \in \Pi$ then for all $\alpha \in R_+$ $(t, \alpha) = \sum n_i (t, \alpha_i) > 0$.

Every Weyl chamber is bounded by exactly r hyperplanes, called the walls of the Weyl chamber.

Definition 2 *The Weyl group* W *of* $R \subset GL(E)$ *is the group generated by* $S_\alpha, \alpha \in R$.

Example 2 *Let $R = A_n$, the root system of $\mathfrak{sl}(n+1)$. Then $\alpha_{ij} = (e_i - e_j)$ just as s_{ij} transposes i and j. So $S_{\alpha_{ij}}$ sends $(t_1, \cdots, t_i, \cdots, t_j, \cdots, t_{n+1}$ to $(t_1, \cdots, t_j, \cdots, t_i, \cdots, t_{n+1}$. So $W = S_{n+1}$.*

Some properties are that $W \subset O(E)$ and $|W| > \infty$.

The first is obvious, the second is because $W \subset S(R)$.

Theorem 1

1. *W acts transitively on the set of Weyl chambers.*
2. *Fix $R = R_+ \cup R_-$. Then W is generated by $S_i = S_{\alpha_i}$ for $\alpha_i \in \Pi$.*
3. *For every root α there exists $w \in W$ such that $w(\alpha) \in \Pi$ so $W\Pi = R$.*

Corollary 1

1. *R can be recovered from Π.*
2. *Say $R = R_+ \cup R_- = R'_+ \cup R'_-$, with simple root sets Π, Π'. Then $\Pi' = w\Pi$ for some $w \in W$.*

Lemma 2 *Fix $R = R_+ \cup R_-$ and let $W' = \langle S_i \rangle$, the group generated by the simple reflections. Then W' acts transitively on the set of Weyl chambers.*

It suffices to prove that for a Weyl chamber C, there exists $w \in W'$ such that $w(C_+) = C$. Let ℓ be the number of hyperplanes seperating C and C_+. Take a segment connecting a point in C with a point in C_+. It will intersect all these ℓ hyperplanes and only them at isolated points. Then $C_1 = S_{\beta_1}(C_0), C_2 = S_{\beta_2}(C_1)$, and so on to $C = C_l = S_{\beta_\ell}(C_{\ell-1})$. ... Proof of theorem from lemma:

1. Obvious
2. Let $\beta \in R$ and C be a Weyl chamber such that H_β is a wall of C. Then $C = w(C_+)$; $H_\beta = w(\text{wall of } C_+) = w(H_{\alpha_i})$ for some $\alpha_i \in \Pi$ and $w \in W'$. So $\beta = \pm w\alpha_i$. Then $S_\beta = ws_iw^{-1} \in W'$.
3. We already know $\beta = \pm w(\alpha_i)$. If $\beta = -w\alpha_i$ then $\beta = wS_i(\alpha_i)$.