Introduction to Lie Groups and Lie Algebras Final Exam

Gabriel C. Drummond-Cole

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1. Let M be the set of $n \times n$ matrices of rank one. We show that M is naturally a manifold, calculate its dimension, and give its tangent space at E_{11}.

   Well, $G = GL(n) \times GL(n)$ acts on M by $x \to^{(P,Q)} PxQ^{-1}$. We want to show that M is a homogeneous space for this action.

   Now, if $x(v_i) = 0$ for linearly independent $v_i, 1 \leq i \leq n-1$, then $\{Q(v_i)\}$ is a linearly independent set of vectors which are killed by PxQ^{-1} so this also has rank at most one. Similarly, if $x(v) \neq 0$ then $PxQ^{-1}(Q(v)) = P(x(v)) \neq 0$ since P has full rank. Also $(P,Q)(P',Q')(x) = PP'xQ'^{-1}Q^{-1} = (PP')x(QQ')^{-1} = (PP', QQ')x$. So this is a well-defined action.

   Write an element $P \in GL(n)$ as $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, where p_{11} is a scalar, p_{12} and p_{21} are $n-1$-tuples, row or column as appropriate, and p_{22} is an $n-1 \times n-1$ matrix. Write Q^{-1} in the same form. Then

   $$\begin{aligned} PE_{11}Q^{-1} &= \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \\ &= \begin{pmatrix} p_{11}q_{11} & p_{11}q_{12} \\ q_{11}p_{21} & p_{21}q_{12} \end{pmatrix}. \end{aligned}$$

   First, we need to show that this action is transitive. Let $x \in M$. By Jordan decomposition over $\mathbb{C}$, we know that either there is a trace to x, in which case there is an eigenvector with nonzero eigenvalue, or every eigenvalue is zero. Then x is conjugate to one of the following:

   (a) $\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

We know that there cannot be a larger Jordan block in the second case or any Jordan block at all in the first case because then the matrix would have rank greater than one.

So this is not exactly the reasoning we want because we may be working over $\mathbb{R}$ and so we may not have full conjugation. So suppose for a moment that we are working over the reals. In the first case, we know that x has a real eigenvector with nonzero eigenvalue and the Jordan decomposition then tells us that the original matrix of x, being conjugate to this symmetric matrix, is diagonalizable over the reals. In the second case we know that there is a nonvanishing vector under the action of x, which we can take as v_2 and extend to a basis $v_1, \cdots, v_n$. Then by adding an appropriate multiple of v_2 to each basis vector we can put x in the second form above by a real change of basis.

Then since (P, P) acting on x gives an arbitrary change of basis, to show that this action is transitive we need only show that the two cases above are in the orbit of E_{11}. But this is immediate. For the first case look to $(\lambda I, I)$ and for the second to (I, I_{12}), where $I_{12} = I + E_{12} + E_{21} - E_{11} - E_{22}$ is the matrix with 1 in the 12 and 21 and $ii,\ i > 2$ positions, zero elsewhere.

Now if we can show that the stabilizer group H of E_{11} under this action is a closed Lie subgroup of G, then M will inherit a unique manifold structure as the cosets G/H.

Now to stabilize E_{11} by an element of G, we must have that $p_{11}q_{11} = 1$ so then these are nonzero so that $p_{11}q_{12} = 0 = q_{11}p_{21}$ implies $q_{12} = p_{21} = 0$. As long as these conditions are met, E_{11} will be stabilized. Then the stabilizer consists of elements of G of form

$$\left(\begin{pmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{pmatrix}, \begin{pmatrix} \frac{1}{p_{11}} & 0 \\ q_{21} & q_{22} \end{pmatrix} \right)$$

This is a subgroup; the zeros and inverse conditions are preserved under multiplication. it is also the preimage of e_1 under the map $G \to k^{2n-1}$ taken by $(P, Q) \mapsto (p_{21}, q_{12}, p_{11}q_{11})$. This map is continuous, so this preimage is closed, so it is a Lie subgroup, as desired.

Then M inherits a manifold structure as cosets. The dimension is the difference in dimension between G, that being $2n^2$ or $4n^2$, depending on whether we are in $\mathbb{R}$ or $\mathbb{C}$, and this subgroup, which has dimension $2n(n-1)+1$ (or twice that). Then the dimension of $M(\mathbb{R})$ is $2n^2 - 2n^2 + 2n - 1 = 2n - 1$, and the dimension of $M(\mathbb{C})$ is $4n - 2$.

Okay, for the tangent space at E_{11}, that was discussed, I think, on September 2nd, when I wasn't there, but I'll take a stab at it. It seems quite reasonable to think that, thinking of M as the cosets or topological quotient of the Lie group G by the stabilizer Lie group H at a point, that the tangent space at that point will be the (at least vector space) quotient of the Lie algebras. Any tangent vector in the stabilizer will be in the kernel of the pushforward of the quotient map; on the other hand by surjectivity of the pushforward in this case the stabilizer should be exactly the kernel. Then restricting to a particular point, we get this quotient relationship. It makes sense to think of $T_eG \to T_{E_{11}}M$ if we think of g mapping to g applied to E_{11}.

The Lie algebra of H will then be subject to the same restrictions as H itself as a subset of $\mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$ with the exception that instead of $p_{11}q_{11} = 1$ we will have $p_{11} + q_{11} = 0$.

So $\mathfrak{h}$ will be, precisely,

$$\left\{ \left(\begin{pmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{pmatrix}, \begin{pmatrix} q_{11} & 0 \\ q_{21} & q_{22} \end{pmatrix} \right) \in \mathfrak{gl}(n) \times \mathfrak{gl}(n) : p_{11} + q_{11} = 0 \right\}.$$

At the level of Lie algebras, then, this is the quotient $\mathfrak{gl}(n) \oplus \mathfrak{gl}(n)/\mathfrak{h}$, so pairs (v, w) where v is a column, w is a row, and $v_1 = w_1$. We can view this row and column as the first row and first column of a pair of $n \times n$ matrices with all other entries zero. This makes sense just on the face of it, because those are really the "directions" you can go: either change the single nonzero vector or move a zero vector to match it.

2. Let G be the group of affine transformations of $\mathbb{R}$, that is, maps from $\mathbb{R}$ to itself of form $x \mapsto ax + b$ for nonzero a. We describe the Lie algebra and exponential for this group explicitly, and discuss semisimplicity, solvability, and nilpotence.

   Let $G' \subset GL(2, \mathbb{R})$ be the matrix group

   $$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^*, b \in \mathbb{R} \right\}.$$

   An element of G' can be denoted (a, b); then $(a, b)(c, d) = (ac, ad + b)$; so this is a subgroup of the subgroup of upper triangular matrices as a subgroup of $GL(2, \mathbb{R})$. In fact, it is the preimage of 1 in that subgroup under the smooth projection map on the 22 coordinate, so is closed in that subgroup, so is closed in $GL(2, \mathbb{R})$. Then it is a Lie group.

   Define a smooth map $G \to G'$ by $ax+b \mapsto (a, b)$. Then $(ax+b) \circ (cx+d) = a(cx+d)+b = acx + ad + b \mapsto (ac, ad + b) = (a, b)(c, d)$ so this is a Lie group homomorphism. It is surjective since (a, b) is the image of $ax + b$ and is injective since $ax + b \mapsto I$ implies $a = 1, b = 0$. So this is a Lie group isomorphism.

   Now we can work in G' instead of G. The Lie algebra for G' will be a subalgebra of $\mathfrak{gl}(2, \mathbb{R})$ and the exponential will be the same. In fact, the Lie algebra will be a subalgebra of the Lie algebra of upper triangular matrices. An upper triangular matrix with c in the ii place exponentiates to an upper triangular matrix with e^c in the ii place so this must be a subalgebra of the matrices of form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Such a matrix exponentiates to $\begin{pmatrix} e^a & e^a b \\ 0 & 1 \end{pmatrix}$ so this is in fact the Lie algebra of G'. Then the bracket, in terms of E_{11} and E_{12} is $[E_{11}, E_{12}] = E_{11}$. So the Lie algebra can be viewed either as this matrix algebra or abstractly as a two-dimensional real vector space with the bracket defined thusly on a particular basis.

   The exponential map, again, takes (a, b) to $(e^a, e^a b)$, that is, the affine function $x \to e^a x + e^a b$.

   So $\mathfrak{g}$ is not semisimple because, as we will show, it is solvable. It is solvable because $[[a, b], [c, d]] \in [\langle E_{11} \rangle, \langle E_{11} \rangle] = 0$. It is not nilpotent because

   $$[E_{12}, [E_{12}, [E_{12}, \cdots, [E_{11}, E_{12}]] \cdots]]] = \pm E_{11}.$$

3. Let V be an n-dimensional complex vector space and let B be a symmetric bilinear form of rank $r < n$. Let $G \subset GL(n, \mathbb{C})$ be the group of linear transformations preserving B.

   (a) We describe the corresponding Lie algebra and find its dimension.

   Let B be the matrix of B with respect to the standard basis by abuse of notation. Then $B^*B_{ij} = \sum_k B^*_{ik}B_{kj} = \sum_k B^*_{ki}B_{jk} = BB^*_{ij}$ so B is normal and thus diagonalizable. Then PBP^{-1} is a diagonal matrix of rank r. By applying the change of basis corresponding to P, then possibly interchanging some basis vectors and rescaling, we get that B has matrix $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ with respect to some basis.

   Then $G = \{g \in GL(n, \mathbb{C}) : g^t B g = B\}$. Write g in block form as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then this says

$$\begin{aligned} &\begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^t a & a^t b \\ b^t a & b^t b \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

   So this says that $a \in O(r, \mathbb{C})$ and $b^t a = 0$. Then by right multiplication by a^{-1} we get that $b^t = 0$.

   Now for invertibility we get that the product of such matrices is of form

$$\begin{pmatrix} aa' & 0 \\ ca' + dc' & dd' \end{pmatrix}$$

   So d must be invertible and then if (a, c, d) is a matrix with the constraints thus far, then $(a^{-1}, -d^{-1}ca^{-1}, d^{-1})$ is an inverse. So the constraints give

$$G = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in GL(n, \mathbb{C}) : a \in O(r, \mathbb{C}), d \in GL(n-r, \mathbb{C}) \right\}.$$

   For the Lie algebra, we then get as usual that

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathfrak{gl}(n, \mathbb{C}) : a \in \mathfrak{o}(r, \mathbb{C}), d \in \mathfrak{gl}(n-r, \mathbb{C}) \right\}.$$

   Then the dimension of this Lie algebra is $\dim \mathfrak{o}(r) + \dim\ c + \dim \mathfrak{gl}(n-r) = r(r+1)/2 + r(n-r) + (n-r)^2 = n^2 + r^2/2 + r/2 - nr$.

   (b) We decompose $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{b}$ where $\mathfrak{g}$ is a semisimple subalgebra and $\mathfrak{b}$ is a solvable ideal.

   Well, here's a shot in the dark. Let $\mathfrak{g}_{ss}$ be $\mathfrak{so}(r, \mathbb{C}) \oplus \mathfrak{sl}(n-r, \mathbb{C})$, where these summands are obtained from the block matrices a and d by subtracting scalars to make them traceless. Let $\mathfrak{b}$ be matrices of form $\begin{pmatrix} \lambda I_r & 0 \\ c & \mu I_{n-r} \end{pmatrix}$.

   So to show that this is the desired decomposition, we have a few steps.

First of all, $\mathfrak{g}_{ss}$ is certainly a subalgebra, and is semisimple as the direct sum of semisimple algebras. That's the easy part. Well, the whole thing is not so hard, but these two facts are obvious by inspection.

Next, for $x, x' \in \mathfrak{b}$ we get that $[x, x']$ is equal to

$$\begin{pmatrix} \lambda I_r & 0 \\ c & \mu I_{n-r} \end{pmatrix} \begin{pmatrix} \lambda' I_r & 0 \\ c' & \mu' I_{n-r} \end{pmatrix} - \begin{pmatrix} \lambda' I_r & 0 \\ c' & \mu' I_{n-r} \end{pmatrix} \begin{pmatrix} \lambda I_r & 0 \\ c & \mu I_{n-r} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\lambda' I_r & 0 \\ \lambda' c + \mu c' & \mu\mu' I_{n-r} \end{pmatrix} - \begin{pmatrix} \lambda'\lambda I_r & 0 \\ \lambda c' + \mu' c & \mu'\mu I_{n-r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \bullet & 0 \end{pmatrix}.$$

Then $[[x, x'], [y, y']] = 0$ for any $x, x', y, y' \in \mathfrak{b}$.

All that remains is to show that $\mathfrak{b}$ is an ideal. This is the same sort of thing.

$$[g, x] = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \begin{pmatrix} \lambda I_r & 0 \\ c & \mu I_{n-r} \end{pmatrix} - \begin{pmatrix} \lambda I_r & 0 \\ c & \mu I_{n-r} \end{pmatrix} \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} \lambda a & 0 \\ \lambda b + dc & \mu d \end{pmatrix} - \begin{pmatrix} \lambda a & 0 \\ ca + \mu b & \mu d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \bullet & 0 \end{pmatrix} \in \mathfrak{b}.$$

4. Let G be a compact real Lie group, $\mathfrak{g}$ its algebra, and V a complex finite dimensional representation. We show that for every $x \in \mathfrak{g}$ its action is diagonalizable, and discuss the truth of this assertion in $\mathfrak{g}_{\mathbb{C}}$.

   Since G is a compact real Lie group every representation of it is unitary. Then for $x \in \mathfrak{g}$ we have $\exp(\rho(x)) = \rho(\exp(x)) \in U(V)$ where ρ denotes both the original representation of the group and the induced representation of the algebra. Then $\exp^{P\rho(x)P^{-1}} = P\exp(\rho(x))P^{-1}$ is diagonal for some P since unitary matrices are diagonalizable, so $\rho(x)$ is diagonalizable.

   This is not necessarily true for $\mathfrak{g}_{\mathbb{C}}$. If $\mathfrak{g} = \mathfrak{su}(2)$ then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. If we let the representation be the standard representation on 2-dimensional affine space, then the matrix of $\rho(e)$ for $e \in \mathfrak{sl}(2)$ is the matrix of e itself, which is certainly not diagonalizable.

5. We show that $V = S^k\mathbb{C}^n$ has the natural structure of an $\mathfrak{sl}(n, \mathbb{C})$-module. We show it is irreducible, find the highest weight, and find the dimension of $V[0]$.

   Well, there is not much to show. S^k can be expressed as a tensor (the symmetric product) so the action comes tensorially as the linear extension of $g : x_1^{i_1} \cdots x_n^{i_n} \to \sum_j i_j g(x_j) x_1^{i_1} \cdots x_j^{i_j - 1} \cdots x_n^{i_n}$ where g acts on x_j as the standard representation on $\mathbb{C}^n$.

   One can check directly that this is a representation; linearity is clear, while for the bracket, this follows from the bracket identity for the standard representation. Then for the diagonal Cartan subalgebra $\mathfrak{h}$ every monomial $x_1^{i_1} \cdots x_n^{i_n}$ is an eigenvector of weight $\sum_j i_j \lambda_j$, where $\lambda_j(h) = h_{jj}$. The monomials clearly span the space so this set of eigenvector/weight pairs must contain all the weights. If we let $t = \sum t_i \lambda_i$ where $i > j$ implies $t_i > t_j$, then t can test for highest weight; Then for the weight written above we have $(t, \sum_j i_j \lambda_j) = \sum i_j t_j \leq \sum i_j t_1 = t_1 \sum i_j = t_1 k = (t, k\lambda_1)$. So the highest weight is at most $k\lambda_1$. Since $k\lambda_1$ is the weight of x_1^k, this is the highest weight.

For $V[0]$, do the following. For a given i, write a polynomial p in $S^k\mathbb{C}^n$ as a polynomial $q_k x_i^k + \cdots + q_1 x_i + q_0$ in x_i. Then $E_{ii}p = \sum_{j=0}^{k} j q_j x_i^j$. For p to have weight zero, this would have to be the zero polynomial, which implies that $q_j = 0$ for $j > 0$. Then p would have to be independent of x_i.

Since this is true for all i, if $k > 0$ the space $V[0]$ contains only the zero polynomial. For $k = 0$ the space $V[0]$ is all of $S^k\mathbb{C}^n$, namely the constant polynomials.

For irreducibility, we apply the same method used in the homework. We know from the semisimplicity of $\mathfrak{sl}(n)$ that this module contains the unique irreducible with highest weight $k\lambda_1$. So then all we need to do is show that every eigenvector in our spanning set can be obtained from x_1^k by the action of our Lie algebra. But this is easy:

$$\prod_{j>1} E_{1j}^{i_j} x_1^k = \frac{k!}{i_1!} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

6. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $(\ ,\ )$ the Killing form, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ the root decomposition. For any $\alpha \in R_+$, let $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ be such that $(e_\alpha, f_\alpha) = 1$; let x_i be an orthonormal basis for $\mathfrak{h}$.

   (a) We show that $C = \sum_{\alpha\in R_+} (e_\alpha f_\alpha + f_\alpha e_\alpha) + \sum x_i^2$ is central in $U_\mathfrak{g}$.

   We know from a theorem in class (November 2) that $(v_\alpha, v_\beta) = 0$ if v_α, v_β are of weights α, β, respectively, and $\alpha + \beta \neq 0$.

   Then I claim that the list $\{\frac{e_\alpha+f_\alpha}{\sqrt{2}}, \frac{i(e_\alpha-f_\alpha)}{\sqrt{2}}\}_{\alpha\in R_+} \cup \{x_i\}$ forms an orthonormal basis for $\mathfrak{g}$.

   What do we have to show? Orthogonality is easy. The theorem says that the Killing form inner product of most disjoint pairs of these is zero. The only cases not covered by it are (x_i, x_j), which is zero because the x_i are orthonormal by supposition, and

$$\left(\frac{e_\alpha + f_\alpha}{\sqrt{2}}, \frac{f_\alpha - e_\alpha}{\sqrt{2}i}\right) = \frac{1}{2i}((e_\alpha, f_\alpha) - (f_\alpha, e_\alpha)) = 0.$$

   For orthonormality, again, the x_i take care of themselves. For the others, we have

$$\left(\frac{e_\alpha + f_\alpha}{\sqrt{2}}, \frac{e_\alpha + f_\alpha}{\sqrt{2}}\right) = \frac{1}{2}((e_\alpha, f_\alpha) + (f_\alpha, e_\alpha)) = 1;$$

$$\left(\frac{f_\alpha - e_\alpha}{i\sqrt{2}}, \frac{f_\alpha - e_\alpha}{i\sqrt{2}}\right) = \frac{1}{-2}((f_\alpha, -e_\alpha) + (-e_\alpha, f_\alpha)) = 1.$$

   So orthogonality shows that the set is linearly independent. It is spanning because the x_i span $\mathfrak{h}$ and the pair for α span $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Then we have, in $U_\mathfrak{g}$, that

$$\left(\frac{e_\alpha + f_\alpha}{\sqrt{2}}\right)^2 + \left(\frac{f_\alpha - e_\alpha}{i\sqrt{2}}\right)^2$$

$$= \frac{1}{2}(e_\alpha^2 + e_\alpha f_\alpha + f_\alpha e_\alpha + f_\alpha^2) - \frac{1}{2}(e_\alpha^2 - e_\alpha f_\alpha - f_\alpha e_\alpha + f_\alpha^2)$$

$$= e_\alpha f_\alpha + f_\alpha e_\alpha.$$

We know that the Casimir element is central; with respect to this orthonormal basis we can calculate its value to be precisely

$$\sum_{\alpha \in R_+} (\frac{e_\alpha + f_\alpha}{\sqrt{2}})^2 + (\frac{f_\alpha - e_\alpha}{i\sqrt{2}})^2) + \sum x_i^2 = C.$$

This shows that C is central.

(b) We calculate the value of C in the irreducible highest weight representation L_λ. So C is central and thus will act like a scalar. We can tell what scalar by looking at Cv_λ. Another proposition (from November 4) says that $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)H_\alpha$, where H_α is the Cartan element corresponding to the dual element α. Then $H_\alpha = e_\alpha f_\alpha - f_\alpha e_\alpha$ so that $e_\alpha f_\alpha = H_\alpha + f_\alpha e_\alpha$. Then

$$Cv_\lambda = (\sum_{\alpha \in R_+} H_\alpha + 2 f_\alpha e_\alpha + \sum x_i^2) v_\lambda$$

$$= (\sum_{\alpha \in R_+} \lambda(H_\alpha) + \sum \lambda(x_i)^2) v_\lambda.$$

The fe vanishes because e kills v_λ. Now, by virtue of the pairing between $\mathfrak{h}$ and its dual, we can replace $\lambda(H_\alpha)$ with (λ, α). Further, because $\{x_i\}$ was an arbitrary orthonormal basis for $\mathfrak{h}$, we can assume that x_1 was the unit vector in the direction of H_λ and that all the others were orthogonal to H_λ. We have $\lambda(x_1) = (\lambda, \lambda/||\lambda||) = ||\lambda||^2/||\lambda|| = ||\lambda||$. Then this scalar becomes $||\lambda|| + \sum_{\alpha \in R_+} (\lambda, \alpha)$.