Introduction to Lie Groups and Lie Algebras December 9, 2004

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Recall that we have a highest weight module over $\mathfrak{g}$. What we proved was the following:

1. Any highest weight module is M_λ/M' where M_λ is the Verma module. It is easy to describe as a vector space; it is $U(\mathfrak{n}_-)$; Every vector in M_λ can be uniquely written as $v = uv_\lambda$ with $u \in U_{n_-}$. It is also this as a U_{n_-}-module, but this does not define the action of the rest of the Lie algebra.

2. For any λ there exists a unique irreducible highest weight module L_λ.

3. $\{L_\lambda : \lambda \in D_+\}$ is the set of irreducible finite dimensional representations. This is the set of weights λ such that $\langle \lambda, h_i \rangle \in \mathbb{Z}_+$.

   We described briefly what this looks like for $\mathfrak{sl}(n)$; it was $\mathbb{Z}_+^r$.

Some of this I did not prove; I did not prove sufficiency of this last condition; more importantly I did not discuss what L_λ looks like. It would be nice to say what is the quotient.

Here is the motivating picture over $\mathfrak{sl}(2)$. Then $L_n = M_n/M_{-n-2}$

$$\begin{array}{cccccccc} v_{-n-2} & \underset{0}{\leftrightarrow} & \leftrightarrow & (f^2/2)v & \underset{2}{\leftrightarrow} & fv & \underset{1}{\leftrightarrow} & v_n \\ & & & & & & & \\ -n-2 & & & n-4 & & n-2 & & n \end{array}$$

What can we learn?

1. Weights of L_n are W-symmetric, where $W = \{1, S\}$ where $S(\lambda) = -\lambda$. This we know in general.

   There is another thing. How are v_n and v_{-n-2} related in terms of weights? The weight of a submodule, i.e., $-n-2$, is symmetric to a weight of M_n, i.e., n. So there is also a symmetry around -1. So this is $s.\lambda = -\lambda - 2$.

2. In the case of $\mathfrak{sl}$ this is quite obvious and trivial. The second point is that you can use this to compute the weight of all the characters involved. Let $ch\ V = \sum \dim\ V[\lambda]x^\lambda$. Then the character of the Verma module are trivial, so this is $x^\lambda + x^{\lambda-2} + \cdots = x^\lambda(\sum x^{-2i}) = \frac{x^\lambda}{1-x^{-2}}$.

$$Ch\ L_\lambda = ch\ M_n - ch\ M_{-n-2} = \frac{x^{n-1} - x^{-n-1}}{x - x^{-1}} = \sum_{-n}^{n} x^i.$$

If we could say that an irreducible finite dimensional module was a quotient of the Verma module by something nice we'd have things easy.

In general this is much harder. We already know we can get it as a quotient but we do not know the submodule.

Theorem 1 *Let $\lambda \in D_+$, so that $\langle \lambda, h_i \rangle = n_i \in \mathbb{Z}_+$. Then $L_\lambda = M_\lambda / \sum M_i$, where $M_i \subset M_\lambda$ is the submodule generated by $v_{s_i\lambda} = \frac{f^{n_i-1}}{(n_i+1)!} v_\lambda$.*

I cannot prove this; it's not difficult but it does take time.

1. The important part is that each of these vectors is that each is a "highest weight" vector in that $e_j v_{s_i\lambda} = 0$. All the directions except one are obvious; the last one comes because of the picture from $\mathfrak{sl}(2)$.

2. If I take $M_i = U_{n_-} v_{s_i\lambda}$ is a Verma module.

3. If I want to relate two of these, they are symmetric not around the origin but around the analogues of -1. If we define the shifted action of the Weyl group by $w\lambda$ around the point $-\rho$, where $\langle \rho, h_i \rangle = 1$. This can be computed rather explicitly. These weights are obtained from λ by the twisted action S_i.

   The possible weights are then on this lattice.

If the story stopped here, we would be very happy.

The natural thing would be to say $ch\ L_\lambda = chM_\lambda - \sum chM_{s_i\lambda}$. The problem is that these intersect. Then I would need a correction term.

Instead of writing this as a quotient, let's write it as a complex. We know $M_\lambda \to L_\lambda \to 0$ and that the kernel is generated by the image $\oplus M_{s_i\lambda}$. So we get $\to \oplus M_{s_i\lambda} \to M_\lambda \to L_\lambda \to 0$. If we knew the weights or characters of all the Ms then we would basically be done. There is a theorem:

Theorem 2 *These can be continued to a long exact sequence. The next term is $\oplus_{l(w)=2} M_{w\lambda}$. The general term is $\oplus_{l(w)=i} M_{w\lambda}$. This is called the BGG resolution of L_λ. That is Bernstein, Gelfond, and Gelfond (there are two Gelfonds). This terminates.*

This will immediately give the character as the Euler characteristic of the resolution. You can write this as $ch\ L_\lambda = \sum (-1)^{l(w)} ch\ M_{w\lambda}$.

To complete the picture I need to tell you the character of a Verma module. I have said that it's easy to compute. It is something similar to what you have on the other side of the board.

First of all, an element of the Verma module is uv_λ, which is $\lambda + wt(u)$. So $ch\ M_\lambda = \sum \dim\ M_\lambda[\mu] e^\mu$ subject to $e^{\mu_1+\mu_2} = e^{\mu_1} e^{\mu_2} = e^\lambda\ ch\ U_{\mathfrak{n}_-}$.

A basis is given by an expression like $\Pi_{\alpha \in \mathbb{R}_-} f_\alpha^{n_\alpha}$.

This is $e^\lambda \prod (\sum e^{-i\alpha}) = \frac{e^\lambda}{\prod 1 - e^{-\alpha}}$. We're almost done.

So what is this thing now, the character of L_λ? It is $\frac{\sum (-1)^{l(w)} e^{w\lambda}}{\prod (1 - e^{-\alpha}}$. This is called the Weyl character formula.

This presentation is historically inaccurate. The BGG resolution was forty years after Weyl did. But this is the best way of deriving this formula, it really tells you what's going on.

I'll do an example in just a second for $\mathfrak{sl}(3)$.

There are plenty of ways to prove this formula; this is the most intuitive but still requires technology. The statement is nice and easy; the proof is not. It has a geometric interpretation in terms of the cohomoloyg of a certain space. So then you get an action on forms, but I couldn't cover that in twelve minutes.

Let me give you one explicit example, over $\mathfrak{sl}(3)$ where the roots are α_1, α_2, normalized so that their norms are $\sqrt{2}$ and their inner product is -1.

Define a dual basis, called the fundamental weight, with the property $(\omega_i, \alpha_j) = \delta_{ij}$. Let's compute a character for a highest weight module. We have $\alpha_i = 2\omega_i - \omega_{2-i}$. I forgot $\rho = \alpha_1 + \alpha_2$. Then

$$Ch\ L_{\omega_2} = \frac{}{(1 - e^{-\alpha_1})(1 - e^{-\alpha_1})(1 - e^{-\alpha_1})}$$

with something more complicated on top. Let's do ω_1.

$$Ch\ L_{\omega_1} =$$
$$\frac{e^{\omega_1} - e^{S_1(\omega_1+\rho)+\rho} - e^{S_2(\omega_1+\rho)+\rho} + e^{S_1S_2(\omega_1+\rho)+\rho} + e^{S_1S_2(\omega_1+\rho)+\rho} - e^{S_1S_2S_1(\omega_1+\rho)+\rho}}{(1 - e^{-\alpha_1})(1 - e^{-\alpha_1})(1 - e^{-\alpha_1})}$$

If you check these things in, you see you don't want to complete the formulas. In general you want a better way. But there is no better way. For $\mathfrak{sl}(n)$ there are combinatorial methods; in general there is no better way. Programs can do it for you but that is the best you can get. In theory it is answered here; in real life it is not answered. Let me stop here. As you know the class ends here. I want a final, but I'll make it a take-home exam. I'll post it on Tuesday, you can give it by Thursday. Next semester I will teach more on infinite dimensional Lie algebras, because the general theory is extremely complicated, so people only look at very simple cases. I will be talking about a bunch of things that naturally occur in physics, but it won't be physics. I'll post the exam on Tuesday.