Introduction to Lie Groups and Lie Algebras
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Let me remind you what we're talking about. We have a semisimple $\mathfrak{g} = \mathfrak{h} \oplus \oplus \mathfrak{g}_\alpha$, and you can write this as $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $n_\pm$ partitions $\oplus \mathfrak{g}_\alpha$ into positive and negative.

If you have a finite dimensional representation V you can write $V = \oplus V[\lambda]$ (often written V_λ) where $V[\lambda] = \{v | hv = \langle \lambda, h \rangle v$ for all $h \in \mathfrak{h}\}$. This is $\lambda \in P(V)$ which as a set has the properties

1. $P(V) \subset P = \{\lambda : \langle \lambda, h_\alpha \rangle \in \mathbb{Z}\} \subset \mathfrak{h}^*$
2. $P(V)$ is W-invariant.

Today I will classify irreducible finite dimensional representations. We tried to generate our representation by some highest weight vector; from it we recover the rest of the representation by applying f.

Definition 1 *A highest weight representation of $\mathfrak{g}$ is a (possibly ifinite dimensional) representation V generated by $v_\lambda \in V[\lambda]$ such that $x_+ v_\lambda = 0$ for all $x_+ \in \mathfrak{n}_+$.*

Generated means that you can get any element in the representation as a linear combination of compositions; $V = U_\mathfrak{g} v_\lambda$.

If you remember, for $\mathfrak{sl}(2)$ we sometimes had infinite sets of vectors which is why you want to allow infinite dimensions. v_λ is a highest weight vector and λ is a highest weight. I require that all my representations have weight decompositions, not necessarily integral. This is unnecessary because you can prove that the representations we need have such a decomposition. But to make my life easier I mean something with such a decomposition.

Theorem 1 *Any irreducible finite dimensional representation is a highest weight representation.*

That's a trivial fact. After all, you only have finitely many possible weights. So all you have to do is choose a weight that is maximal in some sense. Let $t \in \mathfrak{h}^*$ be such that $(t, \alpha_i) > 0$. So it has positive product with all simple, thence positive roots. Take $\lambda \in P(V)$ such that (λ, t) is maximal. There is no question what this means since it is a real number. Then $\lambda + \alpha \notin P(V)$ for all $\alpha \in R_+$. Then $e_\alpha v_\lambda \in V[\lambda + \alpha] = 0$.

The converse, of course, is not true. For some time let's forget about finite dimensional irreducible representations.

Theorem 2 *Any vector in a highest weight representation V can be written as uv_λ for $u \in U\mathfrak{n}_-$.*

So this makes sense since positive weights will kill v_λ and Cartan will fix it. So we need just to get rid of expressions like $e_1 f_1 v_\lambda$; but this is $[e_1, f_1]v_\lambda - f_1 e_1 v_\lambda = h_1 v_\lambda$.

Formally, $U_\mathfrak{g} = U_{\mathfrak{n}_-} \otimes U_\mathfrak{h} \otimes U_{\mathfrak{n}_+}$, or $x = \sum x_-^i h^i x_+^i$ where $x_\pm^i \in U_{\mathfrak{n}_\pm}$ and $h^i \in \mathfrak{h}$.

This is true by PBW; if you have a basis in $\mathfrak{g}$ then monomials respecting an order on that basis form a a basis for $U_\mathfrak{g}$. This proves the theorem directly, if you just order your basis by starting with $\mathfrak{n}_-$, then $\mathfrak{h}$, then $\mathfrak{n}_+$.

Then the theorem follows; if x_+^i is not one then this kills the vector; if it is one then the h^i multiplies it by a scalar so the theorem is obvious.

Corollary 1

1. *All weights of V are of the form $\lambda - \sum n_i \alpha_i$ for $n_i \geq 0$.*
2. *dim $V[\lambda] = 1$.*
3. *λ is uniquely determined by V. The highest weight vector is uniquely determined up to a constant factor.*

The first one is pretty obvious, because the weights you can get from $U_{\mathfrak{n}_-}$ you can write as $\prod f_\alpha^{k_\alpha}$. This gives a basis for $U_{\mathfrak{n}_-}$ so we know how these shift weight; I should also say that every positive root is a combination of simple roots with positive coefficients.

The second one is also obvious. Any element in $V[\lambda]$ should be achievable as $\sum x_-^i h^i x_+^i) v_\lambda$; for any component the x_i^+ must be one for this to work; such a monomial should have weight zero so then x_-^i is one as well.

The third one I will leave to you as an exercise.

There may be more than one highest weight representations with the same weight λ. Let's do some examples.

1. Consider $\mathbb{C}^3$ as a module over $\mathfrak{sl}(3, \mathbb{C})$. The weights are exactly $\epsilon_1, \epsilon_2, \epsilon_3$, where ϵ_i applied to the Cartan element $(\lambda_1, \lambda_2, \lambda_3)$ is λ_i. Let $t = \epsilon_1 - \epsilon_3$; then $(t, \alpha_1) = (t, \alpha_2) = 1$; Then ϵ_1 is clearly highest. We can write $\epsilon_1 = \epsilon_1 - \alpha_1$ and ϵ_3 as $\epsilon_1 - \alpha_2 - \alpha_1$.

2. Consider $\mathfrak{sl})2, \mathbb{C})$ and $\mathfrak{h}^* \cong \mathbb{C}$. Then the positive root is 2. Take a number λ and consider $M_\lambda = \langle v^0, v^1, \cdots \rangle$ so $v^i \in M_\lambda[\lambda - z_i]$.

$$\begin{tikzcd} \bullet \arrow[r, leftrightarrow, bend right, "\lambda-2", "3"'] & \bullet \arrow[r, leftrightarrow, bend right, "\lambda 2 1"'] & \bullet \arrow[r, leftrightarrow, bend right, "\lambda"'] & v_\lambda \end{tikzcd}$$

   Now you have in this infinite dimensional one an inclusion $M_n \supset M_{-n-2}$ and you can take a quotient to find $M_n/M_{-n-2} = V_n$.

I want to generalize the same ideas to the arbitrary case. I want to generalize the analog of this.

Definition 2 *Let $\lambda \in \mathfrak{h}^*$. The Verma module M_λ is the highest weight representation of $\mathfrak{g}$ represented by v_x with relations $x_+ v_\lambda = 0$ for $x_+ \in \mathfrak{n}_+,\ hv_\lambda = \langle \lambda, h \rangle v_\lambda$ for $h \in \mathfrak{h}$.*

You can rewrite this over the universal enveloping algebra. A module generated by a single element over this algebra is $U_\mathfrak{g}/U_\mathfrak{g} \cdot \mathfrak{n}_+ + U_\mathfrak{g} \cdot (h - \langle \lambda, h \rangle)$.

Lemma 1 *Every $v \in M_\lambda$ can by uniquely written as $v = uv_\lambda$ for $u \in U_{\mathfrak{n}_-}$.*

The difference here is uniqueness. Because the module is free the expression is unique.

The proof is the same as before. We already discussed the form it can be written in. Now if I also use uniqueness from PBW I'm done.

The only problem is that these are infinite dimensional. It is about as large as an algebra of polynomials in the appropriate number of variables.

In which sense is it universal?

Theorem 3 *1. Every highest weight module with highest weight λ can be written M_λ/I for some submodule $I \subset M_\lambda$.*

2. For any $\lambda, \exists!$ irreducible highest weight representation L_λ with highest weight λ.

Let me define $M_\lambda \to V$ by $v_\lambda \to v_\lambda$. Since this commutes with the action of $\mathfrak{g}$ then you don't have a choice. This is surjective because you can obtain anything as uv_λ, which is the image of itself. Since M_λ is free over v_λ the morphism is uniquely determined. You have to show that this agrees with all of $\mathfrak{g}$, not just $\mathfrak{n}_-$, but that is easy. This is universal as the largest module for which this holds.

How do you check that for any λ there is exactly one irreducible module with highest weight λ? If you want M_λ/I to be simple then you want I to be maximal. So you want to show that M_λ has a unique maximal proper submodule I_λ, namely the sum of all proper submodules.

The sum of proper submodules in general is not proper. For every submodule, none can contain v_λ, because then they would not be proper. Then for the sum of them the same thing happens.

It doesn't mean it's easy to define the irreducible quotient, but we can try. Can someone tell me what is L_λ? Tanveer? Let's see what happens when $\lambda = n \in \mathbb{Z}_+$? Then $L_n = M_n/M_{-n-2} = V_n$. If $\lambda \notin \mathbb{Z}_+$ then $L_\lambda = M_\lambda$.

Corollary 2 *The irreducible finite dimensional representations coorespond to those L_λ which are finite dimensional.*

Can we describe these? The answer is, yes we can. I don't have enough time to prove this theorem, but I can state it.

Theorem 4 *L_λ is finite dimensional if and only if $\lambda \in P_+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, h_i \rangle \in \mathbb{Z}_+$ for all $i\}$. These are called dominant weights.*

$$\begin{tikzcd}
P_+ \arrow[r, "\subset"'] \arrow[d, "\cong"] & P \arrow[r, "\subset"] \arrow[d, "\cong"'] & \mathfrak{h}_* \arrow[d, "\cong"'] \\
\mathbb{Z}^r_+ \arrow[r, "\subset"'] & \mathbb{Z}^r \arrow[r, "\subset"'] & \mathbb{C}^r
\end{tikzcd}$$

Let me prove one direction of this theorem. Namely, if I consider L_λ generated by v_λ under the action of $\mathfrak{g}$. If I consider now the copies of $\mathfrak{sl}(2)$ embedded in $\mathfrak{g}$, then $e_i v_\lambda = 0$ and $h_i v_\lambda = \langle \lambda, h_i \rangle v_\lambda$. Then consider $U_{\mathfrak{sl}(2,\mathbb{C})_i} v_\lambda$. Then for this to be finite dimensional you'd better have this be $\lambda \in \mathbb{Z}_+$ since it is L_λ. So $\langle \lambda, h_i \rangle \in \mathbb{Z}_+$. Otherwise you generate an infinite dimensional module. If you do this for all i you get this condition.

The trick is to go the other way. I don't have time, let me postpone it til next time. Let me just give some examples.

1. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, then irreducible finite dimensional representations correspond to weights $P_+ = \mathbb{Z}_+$, so these are classified by nonnegative integers.
2. If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then $\mathfrak{h}^*$ is the set of n-tuples of complex numbers quotient the subspace $(1, \cdots, 1)$. Then the lattice of weights is $P = \{(t_1, \cdots, t_n) | t_i - t_j \in \mathbb{Z}\}/\mathbb{C}(1, \cdots, 1)$. Then $P_+ = \{(t_1, \cdots, t_n) | t_i - t_j \in \mathbb{Z}$ and $t_1 \geq t_2 \geq \cdots\}/\mathbb{C}(1, \cdots, 1) = \{(t_1, \cdots, t_{n-1}, 0) | t_1 \geq t_2 \geq \cdots\}$ These are partitions of length $n-1$.

Since my time is up, this is the beginning of a huge combinatorial theory relating this to the theory of $\mathfrak{sl}(n)$. So a lot of representation questions can be answered combinatorially. I don't want to get into this. If you are interested in this, you should read one of the books I have read from the beginning.