Introduction to Lie Groups and Lie Algebras December 2, 2004

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Representations of semisimple Lie algebras.

We have finished with our classification of complex semisimple Lie algebras, and we have an explicit construction for each of them; we have four infinite series and five exceptional ones. What we really want to study are the representations of the Lie algebras.

For the spherical Laplacian we needed only $\mathfrak{sl}(2)$ but for other things we will need something else.

Let $\mathfrak{g} = \mathfrak{h} \oplus \oplus \mathfrak{g}_\alpha$. Now V will be a finite dimensional complex representation of $\mathfrak{g}$. So our goal will be the study of representations. The first thing is that we have an anologue of the result for $\mathfrak{sl}(2)$. There it was graded by the eigenvalue of h. There is a similar result here.

Definition 1 *$v \in V$ is of weight $\lambda \in \mathfrak{h}^*$ if $hv = \langle \lambda, h \rangle v$ for all $h \in \mathfrak{h}$. We denote by $V[\lambda]$ the subspace of vectors of weight λ. We will also write $P(V)$ for the set of weights, that is, $\{\lambda \in \mathfrak{h}^* | V[\lambda] \neq 0\}$. There are only finitely many possible eigenspaces, so this is a finite set.*

Example 1 *Let V be $\mathfrak{g}$ under the adjoint representation, what are the weights?*

Then $P(V) = 0 \cup R$. So $\mathfrak{h}$ corresponds to weight 0 and the others correspond to $\mathfrak{g}_h$.

So for the adjoint representation we get the roots and zero, and the root spaces come with weight 1, and 0 comes with multiplicity equal to the rank of your Lie algebra (dimension of $\mathfrak{h}$).

So we have even a direct sum decomposition; a similar result holds in general.

Theorem 1 *1. Any finite dimensional representation has a weight decomposition:*

$$V = \bigoplus_{\lambda \in P(V)} V[\lambda].$$

2. $\mathfrak{g}_\alpha V[\lambda] \subset V[\lambda + \alpha]$.

3. $P(V) \subset P = \{\lambda \in \mathfrak{h}^* | \langle \lambda, h_a \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\}$.

   P is called the weight lattice.

For the first one you need to show that the action of Cartan is always diagonalizable.

For the second, if you remember $\mathfrak{sl}(2)$, this is a generalization of e and f increasing and decreasing the weight by two.

So let's prove this. The proof is simple. The basic idea is that, let's start with the second one. This is the same argument as for $\mathfrak{sl}(2)$. Say you have $v \in V[\lambda]$ and $e_\alpha \in \mathfrak{g}_\alpha$, then

$$h(e_\alpha v) = ([h, e_\alpha] + e_\alpha h)v = \langle h, \alpha \rangle e_\alpha v + e_\alpha \langle h, \lambda \rangle v = \langle h, \alpha + \lambda \rangle e \alpha v.$$

So that's the easy part. Let me now prove the other two parts. To prove the first one, it suffices to prove that every action from Cartan is diagonalizable. Then commuting diagonalizable operators can be diagonalized simultaneously. But each element of Cartan, well, h_α span $\mathfrak{h}$ as a vector space. So it suffices to show that h_α is diagonalizable. Because h_α is part of an $\mathfrak{sl}(2)$ triple $(e_\alpha, f_\alpha, h_\alpha)$ so you can consider this as a representation of $\mathfrak{sl}(2)$ and then the action of h is diagonalizable.

We can say more than this. We know that the eigenvalues are always integers. So each h_α is diagonalizable with integral eigenvalues. That gives the last part of the theorem.

The proof is based on two things: knowing what happens with $\mathfrak{sl}(2)$ and knowing that every root can be put into an $\mathfrak{sl}(2)$ triple.

Before I go on, let me note that it suffices to just require $P = \{\lambda \in \mathfrak{h}^* | \langle \lambda, h_{\alpha_i} \rangle \in \mathbb{Z}\}$ for $\alpha_i \in \Pi$.

Any element h_α can be written $\sum n_i h_{\alpha_i}$ with $n_i \in \mathbb{Z}$. I'm going to skip a step here if the roots are of different length. But you can note that $P \subset \mathfrak{h}^*$ and with an appropriate basis you can identify this as $\mathbb{Z}^r \subset \mathbb{C}^r$.

You can actually describe wahst is this basis. If P is given by these conditions, take ω_i such that $\langle \omega_i, h_{\alpha_j} \rangle = \delta_{ij}$.

Let's do some examples.

Example 2 1. *$\mathfrak{sl}(2, \mathbb{C})$. So $\mathfrak{h}^* = \mathbb{C}\alpha$, and $\langle \alpha, h \rangle = 2$. Then $P = \lambda \in \mathfrak{h}^* | \langle \lambda, h_\alpha \rangle \in \mathbb{Z}\}$*

So if I write $\lambda = c\alpha$ I get $\{c\alpha | c \in \frac{1}{2}\mathbb{Z}\}$. So the weights here are $\mathbb{Z}\frac{\alpha}{2}$. Recall if we identify with $\mathbb{Z}$ in some obvious way, then the simple root corresponds not to one but to two.

As a side remark we could consider the lattice Q generated by all roots=$\mathbb{Z}\langle \alpha_1, \cdots, \alpha_r \rangle$. How are these related? Here we have $\mathbb{Q} = \mathbb{Z}\alpha$. Generally we have $Q \subset P$. So every root is in an integer weight, but it's not true that any integer weight is a root.

To show this inclusion you need to show that $\langle \alpha_i, h_{\alpha_j} \rangle \in \mathbb{Z}$, so we need to argue that $\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)} \in \mathbb{Z}$. We could say that this is from the root system axioms or from Lie algebra stuff.

Lattice theory says that P/Q is then a finite group. In your homework I ask you to calculate it in some examples.

2. $\mathfrak{sl}(3,\mathbb{C})$. *The simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2 - \epsilon_3$; $\mathfrak{h}^* = \mathbb{C}\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle / \epsilon_1 + \epsilon_2 + \epsilon_3$.*

   What is in this case the weight system? $P = \{\lambda | \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}\}$, which with normalization is $\{\lambda | (\lambda, \alpha_i) \in \mathbb{Z}\}$.

   So the fundamental weights are ϵ_1 and $-\epsilon_3$.

   So now take the simplest representation, the defining representation on $\mathbb{C}^3$. What is the weight decomposition? This is $\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$, which I claim to be an eigenbasis for Cartan. After all, Cartan is the diagonal matrices. So we only have to check what are the weights.

   The weight corresponding to e_1 comes from looking at $\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$. This sends e_1 to $\lambda_1 e_1$ so the weight is ϵ_1. So the weights are $\epsilon_1, \epsilon_2, \epsilon_3$.

   These pictures suggest some symmetries for the weights.

Theorem 2 *For any finite dimensional representation V the set of weights $P(V)$ is W-invariant. Moreover,* $\dim V[\lambda] = \dim V[w\lambda]$ *for all $w \in W$.*

The proof is immediate, actually. The idea is always the same. For $\mathfrak{sl}(2)$ we know the weights are symmetric. Since the Weyl group is generated by reflections, it suffices to check when w is a reflection with respect to some root hyperspace. How do we check it for $w = s_\alpha$? I want an isomorphism $V[\lambda] \cong V[s_\alpha v]$. So I construct an $\mathfrak{sl}(2,\mathbb{C})$ triple $e_\alpha, f_\alpha, h_\alpha$ with respect to this α. Then V is a representation of this copy of $\mathfrak{sl}(2,\mathbb{C})$. So what? Let me assume that $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_+$. In particular $f_\alpha^n : V[\lambda] \to V[\lambda - n\alpha]$ is an isomorphism, this being exactly $V[s_\alpha \lambda]$.

So as a corollary of what we know about $\mathfrak{sl}(2)$ we knot that f_α will give an isomorphism.

Let me finish by this. The easy way to talk about it, instead of looking at weights with multiplicities, write a formal sum with graded dimension

$$ch\ V = \sum_{\lambda \in P(V)} \dim V[\lambda] e_\lambda$$

Here e_λ is a formal variable. This contains information about all the weight subspaces.

Let me also be more precise, by also writing the condition that $e_{\lambda+\mu} = e_\lambda e_\mu$ so as to make it an algebra.

So I write $\mathbb{C}[P] = \{\sum_{\lambda \in P} a_\lambda e_\lambda, a_\lambda \in \mathbb{C}\}$, with multiplication as above.

Then is there an easier way of defining this algebra? It is exactly the Laurent polynomials in one variable if I write $x = e_1$. So this is $\mathbb{C}[x, x^{-1}]$. For $\mathfrak{sl}(2)$ I could write my character as polynomial in one variable. So V_3, which has weights $-3, -1, 1, 3$, what will be the corresponding character. Each of these weight subspaces multiplies by the dimension, and the corresponding generators are x^{-3}, x^{-1}, x, x^3, so the character will be $x^3 + x + x^{-1} + x^{-3}$.

This is a nice way to describe these dimensions, and is sometimes called the character. I won't have time to explain this today. In fact, first of all let me say these characters, since we are Weyl group invariant, this will be Laurent polynomials in many variables (equal to the rank of the group) but the polynomials will be symmetric: $ch\ V \in \mathbb{C}[P]^W$.

So you can show $ch(V \oplus W) = ch\ V + ch\ W$. If you want vectors of weight λ that will just be the sum of those in V and those in W. Then you get this. More interesting is that it agrees with the tensor product: $ch(V \otimes W) = ch\ V\ ch\ W$. If $v \in V[\lambda], w \in W[\mu]$ then $v \otimes w \in V \otimes W$. We have $h(v \otimes w)$ by the Liebnitz rule is $hv \otimes w + v \otimes hw$, so that the weight is $\lambda + \mu$.

This is why I defined my multiplication thusly in this algebra.

You can actually use this to compute various results about decomposition of a given representation into an irreducible. Let me show you how that works for $\mathfrak{sl}(2)$.

Consider V_3 and V_4. The characters are $x^3 + x + x^{-1} + x^{-3}$ and $x^4 + x^2 + 1 + x^{-2} + x^{-4}$.

Multiply these together and you get $x^7 + 2x^5 + 3x^3 + 4x^1 + 4x^{-1} + 3x^{-3} + 2x^{-5} + x^{-7}$. This decomposes as $V_7 \oplus V_5 \oplus V_3 \oplus V_1$.

I'll continue next time and meanwhile I have homework due by the final exam. We are going to have a final exam on December 16 at 11:00 AM. If this creates a serious problem let me know, but I did mention it. I'll try not to make it too difficult, but the homework you should do and bring to the final.