Introduction to Lie Groups and Lie Algebras August 31, 2004

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The website is www.math.sunysb.edu/ kirillov/mat552/ The texts are "Representation Theory: a First Course," by Fulton and Harris, and "Lie Groups and algebraic groups" by Onishchik and Vinberg, which is out of print and on reserve. I won't follow either of these exactly; sometimes I will give references to one or the other of these texts.

I want to have regular homework, a midterm, and a final. Here is the first homework, due in two weeks. I will give a grade, which will take homework and tests into account.

I will require some linear algebra and basic topology, including manifolds, vector fields, and so on.

I'm going to start with a baby example.

Example 1 *Suppose you have m numbers $a_1, \ldots, a_m$ arranged on a circle. You have a transformation which replaces a_1 with $\frac{a_n+a_2}{2}$, a_2 with $a_1 + a_3/2$, and so on. If you do this sufficiently many times, will the numbers roughly equalize? To answer this we need to look at the eigenvalues of A. To get the characteristic polynomial, and then to find the roots, is too hard. But we have a rotational symmetry. We can call this $\mathbb{Z}_n$-symmetry. So $BAB^{-1} = A$, where B is a rotation. This helps us because of the following result from linear algebra: if two operators A and B commute and B is diagonalizable, i.e., that $V = \bigoplus_\lambda V_\lambda$, $B|_{V_\lambda} = \lambda id$, then $AV_\lambda \subset V_\lambda$. We can diagonalize B and see that the eigenvalues of B are the m m^{th} roots of unity. Let ϵ be a primitive m^{th} root of unity. Then the eigenvectors are $(1, \epsilon^i, \epsilon^{2i}, \ldots, \epsilon^{mi})$, where i ranges from 0 to $m-1$.*

So since each of these eigenvectors spans an A-invariant subspace, they are thus also eigenvectors of A. So we can answer the question by looking at what A does to the eigenvectors.

This is the baby version of a real life problem. Say we have $S^2 \subset \mathbb{R}^3$. We have the Laplace operator $\Delta_{sph} : C^\infty(S^2) \to C^\infty(S^2)$. I'm not going to talk about this in detail, it involves extending to $\mathbb{R}^3$ and taking second derivatives. The question is, what are the eigenvalues and eigenfunctions of Δ_{sph}? This describes information about the hydrogen atom. If we try to solve it generally, we run into a difficult differential equation.

So we use symmetry, this time the symmetry of $SO(3)$ acting on the sphere by rotation. This is not a finitely generated group. The second problem is that $SO(3)$ is noncommutative. So the approach before is hopeless, you'll never be able to diagonalize all of $SO(3)$ at the same time, as we could with $\mathbb{Z}_m$. We'll eventually be able to solve this by generating with 3 "infinitessimal rotations" and decomposing into invariant representations instead of eigenspaces.

So how can we deal with these two problems? The trick is that the group SO_3 is more than a group, it is also a smooth manifold.

Definition 1 *A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in some reasonable sense, i.e., multiplication and inversion are smooth maps. A Lie group morphism is a smooth homomorphism.*

I should say what I mean by smooth. For a Lie group, it turns out that C^1 is sufficient to give you real analytic. This is a highly nontrivial result which was one of Hilbert's problems. But the level of smoothness is unimportant here.

1. $\mathbb{R}^n, +$
2. $\mathbb{R}^*, \times$
   $\mathbb{R}_+, \times$
3. $S^1 = \{z \in \mathbb{C} : |z| = 1\}, \times$
4. $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$. A lot of the groups we'll consider will be subgroups of $GL(n)$
5. $SU(2) = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1\}$. This can be seen to be $S^3 \subset \mathbb{R}^4$ by $(\Re\alpha, \Im\alpha, \Re\beta, \Im\beta)$.
6. In fact, all classical groups in linear algebra, such as
   $GL(n, \mathbb{R}), SL(n, \mathbb{R}), O(n, \mathbb{R}), U(n), SO(n, \mathbb{R}), SU(n), Sp(2n, \mathbb{R})$, are Lie groups.

I never said whether my groups were connected as manifolds. So any finite group satisfies this as a 0-dimensional manifold. So we seperate the finite group part and the continuous part as follows. If G is a Lie group, we can denote by G^0 the connected component of unity.

Theorem 1 *This is a normal subgroup of G and is a Lie group itself. G/G^0 is discrete.*

For completeness' sake, it is a group. You have to show that it is closed under the operations of multiplication and inversion. The continuous (inversion) map i must take G^0 to one component of G, that which contains $i(e) = e$, namely G^0. I leave it to you to show that multiplication is closed.
Now, how do you check that this is a normal subgroup? If $g \in G$ and $h \in G^0$, then we must show that $ghg^{-1} \in G^0$. Conjugation by g is continuous will take G^0 to some component;

since it fixes e this component is G^0.
We'll put off for a moment the proof that the quotient is discrete. We're not going to pay much attention to the discrete case.

Theorem 2 *If G is a connected Lie group then its universal cover $\tilde{G}$ has a canonical structure of a Lie group such that the covering map $p : \tilde{G} \to G$ is a morphism of Lie groups, i.e., that it agrees with the group structure. Then* $\ker\ p = \pi_1(G)$. *You can prove this.*

Definition 2 *A Lie subgroup H of a Lie group G is a subgroup which is also a submanifold.*

Theorem 3

1. *(easy) Any Lie subgroup is a closed submanifold*
2. *(hard) Any closed subgroup of a Lie group is a Lie subgroup.*

A torus is a Lie group, since the product structures of groups and manifolds respect one another. So if the torus is G, and H is $\mathbb{R}$, then the map $H \to G$ given by $t \to (t, \sqrt{(2)}t)$ is not a submanifold.

Corollary 1

1. *if G is a connected Lie group and U is a neighborhood of e, then U generates G.*
2. *Say $f : G_1 \to G_2$, G_2 is connected, and $f_* : T_eG_1 \to T_eG_2$ is surjective, then f is surjective.*

Proofs:

1. Let H be the subgroup generated by U. Then H is open in G. That's because multaplication of $h \in H$ by U yields a neighborhood of h in G. Since it's an open subset of a manifold, it is a submanifold, so that it is a Lie subgroup. Therefore it is closed, and is nonempty, so is all of G.
2. Given the assumption, the implicit function theorem says that f is surjective on some neighborhood U of e, and then because U generates G_2, we can pull back to preimages to see that G_1 covers G_2.

That's all I wanted to talk about today.