No class on Friday. The next class is next Monday.

So let's consider Riemann cutting procdedure. We have a surface and last time we sketched an argument that each surface with $g > 1$ can be constructed out of a non-Euclidean sheet, in fact, with $6g - 6$ free parameters. So let's consider this a bit more. The first step is to choose a closed curve along which to make a cut. Allow it to tighten into a geodesic. When you cut along it, you get two boundary components and it looks locally like a half-space at each. More generally we can glue together two half-spaces to get a surface.

So for the first choice take the shortest nonseparating closed curve. Later I'll discuss the choice of a shortest among all such. The next cut is the shortest nonseparating arc. All following cuts are the shortest nonseparating arcs.

What we observed last time is that these angles are always right angles when you pull along shortest curves. Then when you cut it open the first two cuts create four sides, and then each time you create four new sides and you get $4 + 4(2g - 2)$ sides. This is $8g - 4$ sides. If you're building a right-angle gon, then if you put all but the last three sides in, the last three are determined by the rest. So there's really $8g - 4 - 3$ free parameters when you try to go backwards. Then gluing back together you need all glued lengths equal, which gives you $2g$ constraints. Then at the last moment you get an extra twist parameter, for gluing two circles, which gives you $6g - 6$.

Now, I'm going to come back to, I mean, we counted the number of combinatorial types. It was $3 \cdot 7 \cdots 4g - 5$. This was the number of combinatorial parameters. What would be nice, I won't get to it today, is let $\mathscr{M}_g$ be the set of all non-Euclidean surfaces of genus g up to congruence. Is there anything wrong with this sentence? We want to make that a category. But, anyway, you know the story of Frege and Russell? He's developing the numbers, the integers, you know, you have, one two three dot dot dot, so what is an integer? So Frege said take equivalence classes of finite sets. He started his book with "Take the set of all finite sets." Then Russell wrote this letter. He said, "Never has someone's life's work been demolished in so elegant a fashion, but mine has." So you have to take the equivalence classes and prove that the collection is a set.

So you put them as submanifolds in a Hilbert space. Well, that doesn't help Scott. You can take one manifold cross any conceivable notion. This is a use of category. But the morphisms $A \to B$ form a set. But the objects don't form a set. In fact you call it a small category if the objects form a set.

So let's consider these isomorphism classes, these congruence classes of surfaces. It's kind of a research problem to take this set and divide it up into subsets with certain properties. I'll discuss this more later.

Now, one interesting thing; we get something from this picture of a different sort. Let's go back to topology. I mean, roughly speaking, we've divided this set up into $3 \cdots 4g - 5$ different regions, and how do they intersect?

So what I want to do is mark all the points where the corners are. Let's suppose there are no overlaps among corners. Now how many points are there? Each time you cut after the first, you add a point. You get $2(2g-1) = 4g-2$ vertices. This figure that we get is called a graph, and all these little pieces, the cuts break up into pieces called edges. How many edges do we have? This is called a trivalent graph. At every vertex there are three edges. If you pull them apart slightly, this is an important, very useful argument. The number of pulled-apart points, $3V = 2E$. So there are $E = 3/2V = 6g-3$ edges.

For any graph, you define the Euler characteristic of the graph to be $V - E$, so here $4g-2-6g+3 = 1-2g$. The interesting thing about the Euler characteristic is that it just depends on the figure and not on the decomposition into edges and vertices.

If we added a vertex, then we would break an edge in two and not affect χ, i.e., $\Delta\chi = \Delta V - \Delta E = 0$. We want to extend this Euler characteristic to two-dimensional generalizations of graphs. I want to define what I mean by a $2-d$ cell complex.

Definition 1 *A 2D cell-complex is a graph unioned with a collection of sewed on n-gons.*

So a surface is such that each edge is covered twice. So there are 15 edges in a graph for a genus 3 surface. We're gluing on an $8g-4$-gon, which is 20, so that isn't working out very well. So the graph is the 1-skeleton, if you know those words. I'm going to give an example in a minute of more complicated 2-complexes. If you have a 2-dimensional cell complex and each edge has two sheets, then this implies it's almost a surface. Things can still meet at vertices which are singularities. For surfaces we have this result where the surface is determined by its genus. So it's easy to describe cell complexes which have two sheets at every edge. You just take surfaces and identify finite sets of vertices.

In general 2-complexes are complicated to classify. In fact, they're not classified. It's not even an algorithmic problem to classify them. Both up to homeomorphism and up to homotopy these are hard to classify.

This means you can't write a computer program to do this. Let me say, the Euler characteristic is, certain aspects of these theories are algorithmic. For a $2D$-complex the Euler characteristic is $V - E + F$. These definitions were floating around long before Riemann was working. This is the original definition, probably by Euler. Actually probably the first traces of it are [unintelligible].

So here's a property: χ is unchanged by subdivision. We have a graph, with faces glued onto it. If you add a vertex in the middle of an edge, the Euler characteristic doesn't change. If you add an edge you don't change the number of vertices but you add one each to the numbers of edges and faces, again not changing the Euler characteristic. So you can get down to where all the faces are triangles. So there's a number attached to these one and two dimensional spaces not dependent on how you chop it up.

So let's compute it for the closed surface I had. I had $4g-2$ vertices and $6g-3$ edges with one face. I glued on this one $(8g-4)-gon$. This gives me $(4g-2)-(6g-3)+1 = 2-2g$.

So the Euler characteristic, it's very algorithmic, you just count the number of pieces in each dimension. Probably if you went to the internet and found how many hits there are for concepts in algebraic topology, this would probably win because it's the simplest to use.

For example, if you have two intersecting complexes A and B you have $\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B)$. So then, for example, one can use this to prove this formula of Riemann, that the genus is $1 - \#sheets + \frac{1}{2}\#defects$. You have the two-sphere, and then over it some number of sheets, and then points where they come together. The cardinality of the generic fiber is the numbr of sheets. So I formally add a point where they drop together. So if I add a bunch of points to make the number of points over every point equal to the number of sheets, then the new space upstairs is $d\chi_{downstairs}$ where d is the number of sheets.

Let me write this down, the number of vertices upstairs is d times the number downstairs. The same is true for edges and faces. So then you get that the Euler characteristic upstairs is d times the number downstairs. So that gives $2d = \chi_{surface} + defect$. So this $\chi_{surface} = 2g - 2$ is our new definition of genus. We know it's consistent because we showed how this worked with the cuts. I solve for the genus and get $g = 1 - d + \frac{1}{2}defect$.

This is such a nice invariant, when I was a graduate student I wondered if there were any other invariants of subdivision. So that's a thought problem.

Exercise 1 *a. Are there any other invariants of subdivision that only depend on V, E, and F? (for 0, 1, and 2-dimensional spaces.*

b. Show that $1 - \chi_{connectedgraph}$ is a nonnegative integer. Guess what the integer is qualitatively.

So part of this is to say, what does that mean? Formulate a mathematical statement.

This is how the Atiya-Singer theorem came about. They had an integer, and so they figured it was the index of an operator, so they invented the operator. Integers, especially nonnegative integers, are really important.

Tom? Do you want coffee, Tom?

For $3D$, you have cubes, like rooms, but if you pass through a wall, there may not be any room, or there may be more than one room. If you have a space built out of other things, you can always break it down into cubes. A tetrahedron is the union of four cubes.

A $3D$ cell complex is a psuedo manifold if each wall separates exactly two rooms. So this is analogous to our graph on the surface. The three dimensional analog is that for each wall there's two room. The boundary of the hotel has only one room next to a wall.

And then you can study how many rooms are incident on an edge. Any time you cross a wall there's another room, but then you can start going around an edge, and by finiteness you'll come back eventually. So how many cycles of rooms are incident on an edge? So assume further that for each edge there is only one cycle. Then around the edge it just looks like space divided into sectors.

Then we go to the vertices. We have this crazy abstract hotel, and for every one of the six walls of each room, there's a room on the other side. We assumed that at an edge, going through all the rooms that have this edge, they form one cycle. So for the vertices, there are triangles fitting together and you get some kind of surface around each vertex. There are little triangles that have to fit together by my edge assumption.

So let's say everything in the room is orientable, so that you could orient the triangles and get an orientable surface. Then you make a final assumption, that the Euler characteristic at each vertex is equal to 2. At a wall, saying the number of rooms coming off is a discrete set, that says it's two. These are three Euler characteristic statements, well, not quite. Then, so, first, this is algorithmic, so if I gave you a cubical complex on a computer, you could have the computer check. A computer could do this in two seconds. So then, if this is true, if the Euler characteristic is 2, this has to be the 2-sphere according to Riemann, so that this is an algorithm that tells you whether it's actually a manifold. If you have a two-dimensional one, if it has three sheets coming together it's not a plane, it has a different topology.

So I need to say that the surface is connected, the surface around a vertex. So this is, it's rather remarkable that we have this notion of manifold in topology. This means that locally it's like Euclidean space. At each wall you have to fit two together. At each edge you need a cycle, and at each vertex you need surfaces like 2-spheres.

It turns out there is no algorithm, for dimension five or higher, to tell whether a cell complex is a manifold.

Poincaré's last paper was about writing a counterexample for the cognates of these algorithms in four dimensions. There was a deep result in the 90s that there is an algorithm.

So next class is a week from today.