So now I'm going to start backwards. For $g = 2$ you get a 12-gon. For genus g it's an $8g - 4$-gon.

Let's imagine going backward, meaning that we would undo the cuts by gluing two sides together. For example I might be able to glue my 12-gon to get a cylinder whose bounding circles were a 3-gon and a 7-gon. I want to think of these as right angles so that when I join these together I can erase the vertex. So really it's a 2-gon and a 6-gon.

(Dennis leaves looking for his son)

Okay, so I'm going to do another gluing and get a torus with a hole in it. Now after erasing the angles where things met I get a 4-gon. Now I can only glue opposite sides. I can erase the gluing line, and then you get two smooth circles and then gluing those together you get a surface of genus 2.

In a moment I'm going to do it geometrically and then it really will be a right angle. This also tells us how we can do this backward. The labelings come in when you identify edges and when you erase corners. There's some labelling that you can put on this that will correspond to the cuts you've got. So the exercise has to be modified. Instead of drawing the two ways, you need to do three.

Exercise 1 *Find the three Riemann cutting systems for a surface of genus two.*

So I miscounted; let's go back and do that again.

Proposition 1 *The number of ways to do Riemann cutting on the surface of genus g is* $\prod_{k=2}^{g}(4k-1)$

Proof is by pictures. The first cut introduces two circles, the second splits the circles to a 4-gon, so far unique up to dihedral symmetry and relabelling of the eventual n-gon.

On the 4-gon you can cut three ways; from a side to itself, from a side to an adjacent side, or from a side to an opposite side. Then you add four vertices uniquely up to combinatorics.

So if you had a 12-gon, the next choice would be from a side to a side n distant from it, where n ranges from 0 to 6.

Now I wanted to have a little geometry interlude. This fits nicely with a geometric picture which Riemann didn't, but Poincaré knew. Somehow this gives you a nice way in. What kind of geometries could these surfaces have? They could have the Euclidean geometry, like the torus does. The sphere has spherical geometry. And then there's one more geometry, a homogeneous geometry. Both of the two I've named are homogeneous; things look the same at all points in all direction. So spherical is positive curvature, the flat geometry is the zero curvatures, and in every dimension it turns out that there can be only three homogeneous geometries.

If you made a pringle potato chip out of leather, you could rotate it around every point. It's amazing that this geometry exists; it's hyperbolic geometry and it has negative curvature.

In spherical geometry you can have a 3-gon with right angles; in flat geometry you can have a 4-gon with right angles.

(Interlude on polar bears)

In hyperbolic space you can have n-sided figures with right angles. One model for this space is the upper half-plane, with geodesics the vertical lines and the half-spheres with boundaries on the boundary of the half-plane. Then you can easily construct these things. In the disc model you do it by deforming an ideal regular n-gon with 0-angle continuously to a very small one which has angles approximating Euclidean angles. So if we begin with a 12-gon that we want to paste together smoothly, it has to be a hyperbolic 12-gon.

So one way that this differs is that instead of parallel lines being isolated, they exist in intervals; i.e., deforming parallel lines by a small enough angle keeps them parallel.

Anyway, this gives you the following picture. If you're building in this geometry an n-gon with right angles, go to where you put in all but 3 of the sides. That determines the final three sides. If that doesn't work then you've overextended, like you have a company that you've driven too far in debt. So the number of parameters is $n-3$. The theorem is that we can build a genus g surface by geodesic right angle gluing in the number of ways we'll get to. So so far we have $8g-4-3$ lengths that can be chosen in the original $8g-4$-gon. So I lose a parameter when I try to identify two sides. The next gluing will involve identified sides. So eventually there will be $2g$ gluings each of which will kill a parameter. Let's leave off the last one, so that we've killed $2g-1$ parameters and I have two smooth circles, so I subtract one more. But now gluing a circle to a circle you get a twist parameter for the gluing.

So the total number of degrees of freedom for this construction is $(8g-4)-3-(2g-1)-1+1 = 6g-6$. This is the dimension of the space of Riemann surfaces of genus g for $g > 1$. We can add the comment that all of them occur this way. This is true by cutting.

So say we have this closed surface. It can't be embedded in space because you bring it tangent to a surface and then it has positive curvature since it's all on one side of the plane. Maybe you can embed it in 4-space, I'm not sure. Hmm. Do any of you know this book? Geometry and the Imagination, it's a layman's book, but with more geometry in it than most mathematicians know. It's by Hilber-Cohn Vossen.

So we want to look for geodesics and choose a shortest one. We cut it open along this shortest one; this gives us a space with good geodesic boundary. Then we look for the shortest curve from boundary to boundary which intersects at right angles. Again we get a geodesic boundary with right angle corners. Now we make another cut, taking the shortest one, and we don't know where that will be, but eventually that gives us an $8g-4$-gon. It is locally hyperbolic.

So I can do this for Euclidean geometry on the torus. What are the number of parameters there? You start with a rectangle and then glue back to a cylinder. Then you glue back with a twist. That's three parameters but you lose one because of the scale parameter, because the curvature is zero.

That's all I wanted to say today, but I have another (optional) problem.

Exercise 2 *Give a combinatorial interpretation of this number* $3 \times 7 \times 11 \times \cdots$ *If it was all the odd numbers it would be the number of ways of pairing* $2n$ *elements. Does this number have such an interpretation.*

Remark 1 *If we start from any surface, there's usually a unique shortest geodesic. Then the next one is unique generically and so on.*

So the space of all such surfaces is divided into regions which have a unique choice. Then the number of regions is this combinatorial number.