So some of the stuff that I'm going to say to you is not available in easily available texts. So I'd kind of be interested in a notetaking committee being formed. So this person seems to be taking notes, and Scott is interested by definition, so anyone interested in dontating notes should give them to Scott or, what's your name, to Gabriel.

Scott, have you collected homework yet? No? Well, give everything you have to Scott.
So we're in the midst of studying this construction of Riemann, and I'm sort of using it to lead us into some concepts of topology.

Over the weekend I noticed it also has a nice interpretation in terms of Galois theory, so we get two theories for the price of one.

So we're starting from an algebraic equation $f(z, y)=0$, which cannot be factored into two other polynomials (i.e., $f=f_{1} f_{2}$ fails except for constant $f_{i}$. I'm supposing it has the form $y^{n}+a_{n}(z) y^{n-1}+\ldots+a_{1}(z)$, and that for some $z$ it has $n$ distinct roots. We're working in $\bar{C}$ which is the plane with an extra point at $\infty$. Then I want to remove a small disc around all these points and then study the set of $(z, y)$ such that $f(z, y)=0$ where $z$ is outside these small discs.

Then I shrink these disks to points and complete the picture. So I don't know what happens over infinity. If I cut out a neighborhood of infinity, it's a region whose complement is bounded. Sometimes I want to think of them as big, sometimes as small. For every point elsewhere in $\bar{C}$ I get $n$ distinct roots. The basic assumption, which really requires proof, but for our discussion, well, is the following. If I move the point $z$ a little bit, then the $n$ roots move continuously.

Now does anyone think this is trivial, false, or true and nontrivial? Those are the possibilities. Most of us think it's true. It can't be completely trivial, because if I hadn't made the assumption that it was monic, this wouldn't necessarily work as the first coefficient went to zero.

I don't have an illuminating proof. I've given you exercises and thought problems. Were they thought problems? Well, this is an exercise.

Exercise 1 Show that the zeros of $y$ vary continuously with $z$ outside of a neighborhood of the forbidden region.

I think that the Galois group of this polynomial over $\mathbb{C}(z)$ is $S_{n}$. That'll be a homework problem for me, maybe.

Now, what was I talking about? I was saying something else. I don't remember what I was saying. So Abel and Galois worked out this thing that you've all heard of, that you can work out a formula with radicals for the roots of a polynomial in terms of the coefficients if and only if the galois group is solvable.

So what we should think of, lying over this, is that over each point we have $n$ distinct roots. If we move the points a little bit, then the roots in the $y$ plane move continuously.

We get this idea that there are $n$ little discs above every little disk in the $z$ plane. So now let's look at what happens over one of these little circles. As you move around one of these circles, the $n$ distinct points come back with some sort of rearrangement.

So now the question is, how many different pictures (permutations) are there? How about $n!$ ? That's the number if I choose a labelling. How many different pictures are there if they're unlabeled?

With two you get two; with three you get three. With four, five; with five, seven. So the general answer is the number of partitions of $n$, or the number of conjugacy classes of $S_{n}$. This is a good exercise, I gave you the solution.

Exercise 2 Show that the number of partitions of $n$ is equal to the number of conjugacy classes of $S_{n}$.

A hint for this exercise, is that if you have a partition of $n$ points, write it as a cycle in $S_{n}$.
So if you had $y^{2}=a(z)$ you got the two roots to interchange. When you had $y^{3}=a(z)$ you got a cyclic permutation of order three.

Riemann's construction, when you go around one of the branch points, what lies above this circle, is this union of circles. In general the number of circles is the number of partitions of $n$ coming into this conjugacy class. Then you shrink each of these circles to a point and put a disc over them, a $z^{3}$ kind of disk. We also do that for the little circle around infinity, as we shrink in around it. So if we have $n$ sheets above a generic point, we'll glue in discs according to the number of partitions you get as you go around the circle. Locally it's just like taking $n$th roots of something.

So that's how we fill in here and then also at infinity. So we're really using the locus of this equation to describe this construction. Using these basic assumptions of continuity, you see you have to look a certain way around a problem, and then you just complete it with those points.

It turns out the locus is not necessarily doing this. You could have two disjoint discs, but Riemann's construction puts in two discs "tangent" at a point. If you look at $y^{2}=z^{2}-\epsilon^{2}$, as $\epsilon$ goes to zero, you get two complex lines, so that the locus looks one way and the Riemann construction looks differently.

You can treat infinity like a finite point by using some projective words. In the projectivization, if the infinity points have this kind of problem, you just pull them apart. Actually, there was a famous question in algebraic geometry, the famous thesis of ${ }^{* * *}$, that you can pull the singularities apart in higher dimensions, at least for fields of characteristic zero.

So now let's think about, that's how it looks at this end, how does it look in the "good" region? you can think of the bounded region as a roadway with the holes really big. You guys are too young to know about slotcars. Imagine you're building a slot car track. So you have $n$ slot car tracks that you have to connect with these ramps. How do ou describe it? Again we use these permutations.

When you buy it at Toys ' $R$ ' Us, it comes to you cut up to pack it better. Is that simply connected now? It's very expensive to make the cuts so we want to make the minimal number. So once you've done that and made your slotcar roadway simply connected, you have no interesting loops. We have four cuts in this particular example.

So then if I look above the roadway on the $n$ sheets, I can find $n$ copies and an instruction sheet to connect these together. So this is hard to do in your bedroom because it is four dimensional. So for every one of these cuts you get $n$ more permutations telling you how to attach things.

Now how many ways can this be done? If you label the ramps, then you get $(n!)^{4}$. I used to have a big slotcar track in my office at MIT, on ropes. I never had time to play with it, my students played with it.

So how many pictures without labeling? Is the answer the partition to the fourth? Just use psychology. The answer is no. It's the number of simultaneous conjugacy classes. This is the number of conjugacy classes of $S_{n}^{k}$ under the action of $S_{n}$. Even if $k$ is two, you get the invariant, the subgroup $\left\langle\sigma_{1}, \sigma_{2}\right\rangle$ in $S_{n}$, which is a big number already, the number of these. So this is a hard problem. There was a guy here recently who was relating this to all kinds of things.

So this is simple in some sense, but then complicated in some other sense. It's often easy to construct a set of objects, but then very difficult to classify them. The difficulty is the projection map. But if we ignore the projection map and just take this to be a surface with boundary, then it follows from what Riemann did that there are two integer invariants which classify this completely. Upstairs if you ignore the level structure, it can be thought of just removing a certain number of discs from a surface of a certain genus. So that's a complete invariant, when we get into topology. One way of looking at this is basically impossible, it's all of finite group theory. The other way you just get the genus and the number of boundary components. If you have a closed surface and you remove discs in different places, that doesn't matter. Above each circle you remove a number of boundary components equal to the number of cycles.

So depending on how you look at this it's either impossible or easy.
I think I'm going to skip the Galois theory remarks. So if we take one determination of the root above some point, (we'll make the holes small again) we get a completed surface of $n$ sheets around the punctured Riemann sphere. The closed thing is completely determined by the genus, which is, for connected surfaces, $1-\#$ sheets $+\frac{1}{2} \#$ defects,

You adjoin $y$ as a new element of the field, in Galois theory, over $\mathbb{C}(z)$. Then we adjoin this new element $y$ which is well-defined on this Riemann surface by this construction. Then this is a geometric picture of a finite extension of the field of rational functions. This really helped me understand this book on finite field extensions.

Then we might want to take another extension, so that we get a branching over a Riemann surface instead of over a sphere. Then you get ramified and unramified field extensions, where the unramified have no branch points. Over the sphere, they'll all be ramified.

Now let's change gears. I want to discuss the ways in which, well, Riemann's construction II, making a closed surface simply connected by making a minimal number of cuts.

A cut on a closed connected surface is a nonseparating closed curve.
A cut on a connected surface with one boundary component is a nonseparating embedded arc with its boundary in the boundary of the surface.
A cut on a connected surface with multiple boundary components is a nonseparating arc with its boundary in two boundary components.

We start from a closed surface, and intuitively I'm going to think it has some sort of length. I have scissors so I want to make these as short as possible. So you cut the genus two surface to get a twice punctured torus. The next cut connects the holes to give the punctured torus.

The boundary component is naturally a quadrilateral, i.e., has four corners. We'll remember that for fun. So then cut next on the punctured torus through the handle to get a cylinder, each of whose ends has four boundary components. Cut this and get a 12-gon with identified sides.

So two cuts kills a genus. The number of cuts to get to something simply connected is twice the genus (plus the minimum of 0 and one less than the number of holes). The simply connected surface is naturally an ( $8 \mathrm{~g}-4$ )gon. You don't cut at a vertex because this is more natural. You add four for the first two, and then 4 more each of the remaining $2 g-2$ cuts.

So call these cuts, say, highways 1,2 , and what's the guess for the next one? 4? No, good guess, powers of 2 , but we'll call it highway 3 . So we get a highway system from the surface. This is a graph with some highway signs. It's trivalent, i.e., all the intersections are $T$ intersections, with the graph having $6 g-3$ edges and $4 g-2$ vertices.

I was trying to get to the exercise.

Proposition 1 There are at least $2^{g-1} \prod_{i=1}^{g}(2 i-1)$ ways of doing the Riemann construction.

Exercise 3 When $g=2$ this is 2 . What are they?

This may be off by one. Anyway, the idea is to look at the number of sides of the boundary components.

