I recommend reading chapter 0 of Hatcher.

Now, I want to go over something that came out at the end of the last class, which is whether the Riemann surface is connected. We'll look at the example $y^2 = (z+\epsilon)(z-\epsilon)$, as ϵ goes to zero. Generically you have two roots of this equation, so that there's a two-sheeted cover, and no branch at infinity We picture the Riemann surface by gluing two slit spheres together along the slits. As ϵ tends to zero, the slit shrinks to a point. In the limit, as you go around one of the points, you go around both of them, so that nothing is changed; it's like infinity in that you stay on the same sheet. So $y^2 = z^2$ produces two disjoint spheres, rather than spheres joined at a point, say. We have the definition of a construction, and this construction yields two spheres.

Now this equation is reducible, i.e., factors as $(y-z)(y+z)$. You imagine that if you have $F_1(\ )F_2(\ ) = 0$ then you get a picture for each factor. Some people have asked me also whether the Riemann surface is the same as the completion of the projectivization in the complex plane. This equation gives you two lines which touch at a point, but that's not the Riemann surface, which can have no singularities.

So we have this construction due to Riemann. It always produces a surface over the z-plane, completed at ∞.

So to simplify the discussion, let's imagine we're in the case where you can't factor these equations. Then check if we get a connected surface.

So let's see, if we havethe equation $y^2 = (\quad)(\quad)(\quad)$ or $y^3 = (\quad)(\quad)(\quad)$ (this was one of the exercises, to make a picture of this). Now in all these examples you separate the variables. This has a symmetry, because all of the roots look the same. So these covers have a symmetry; we call them Galois coverings. So this is kind of the extension of Galois theory to larger fields, like $\mathbb{C}(x)$. So if you have the equation $y^n + a_{n-1}(z)y^{n-1} + \ldots + a_0(z) = 0$, this is kind of a geometric picture of the field extension. Some fields have a sufficient group of automorphisms for a splitting field and some don't.

So not all coverings have a symmetry group that permutes the sheet. The definition of a Galois covering is that there is a finite group of symmetries that transitively permute the sheets. So I wanted to construct some non-Galois coverings.

So maybe since I've said all that, let me look at the equation $y^2 = z^3$ just for a second. Formally, you could write this as $y = \pm z^{3/2}$. This is like $y = (\sqrt{z})^3$. So when you look at the Riemann surface for that, the three is unimportant and you just get a level cover of the sphere.

Near the origin, in $\mathbb{C}^2$ (the reference is Milnor's orange book "Isolated Singularities...," one of his great books). That's how I learned topology, from his notes. Anyway, if you look at how a 3-sphere around the origin intersects this equation, you get a curve in the three-sphere which is actually the trefoil knot. So this is embedded in $\mathbb{C}^2$ as the cone of the trefoil in 4-space. So the Riemann surface completely ignores this information, the cusp singularity information, that is in the embedding, and looks at the locus as an abstract space rather than an embedded one.

So if I want to solve this equation abstractly, I just adjoin $\sqrt{z}$, which gives the double covering of the Riemann sphere, which is just the Riemann sphere.

The knots that you get in general here are completely understood, although they form an interesting class.

So there's an interesting theory I want to go into a bit more. So let's consider what are called simple coverings of the completed z plane, where every branching is just a transposition. So what I'm saying is that generically we have n sheets, and when you go into a branch point, all that happens is that some pair of them cross. Then when you go around, all that happens is that two of the n roots are interchanged. We're saying the zeros of the derivative are simple.

There are interesting things to say about these, and every other example is a degeneration of this example, so they should be studied in their own right. I'm going to decompose the extended z-plane as everything inside and everything outside some circle. I'll have two singularities and three sheets. To describe it, I slit out to the boundary from each singularity and cut out to the boundary, and then I just need to know how to glue them back up. In one case we choose the permutation (12)(3); in the other I choose (1)(23). So what happens if I go around the entire circle. We go around the composition of these two cycles, namely (123). So the boundary wraps around three times to get back. Then some surface fills in along that boundary.

So take two copies of this. Then take the base and think of it as half the sphere, and glue them together along their boundaries. Then I get something which has, well, let's compute the formula. Anybody here heard of the Vlaschke products? This is a picture of a Vlaschke product. The formula is the genus is $1 - \#$ of sheets $+1/2\#$ of defects. This is $1 - 3 + 1/2(4) = 1 - 3 + 2 = 0$ So this is the sphere. This is a non-Galois covering of the sphere by the sphere.

The Vlaschke product takes z to $\frac{z-a}{1-\bar{a}z}$. Take z to $\frac{z-a}{1-\bar{a}z}\frac{z-b}{1-\bar{b}z}\frac{z-c}{1-\bar{c}z}$. Each factor maps the unit disc to itself, so the product does as well. The map of the boundary is of degree 3 because $a,\ b,$ and c tend to zero continuously, so it looks like $z \to z^3$.

Alright, so you can also do, where you take $n-1$ singularities over n sheets, and then over the i branch point you interchange i and $i+1$. Everything else is the identity. So then the composition is $(12)(23)\cdots((n-1)n)$, which is the cyclic permutation around the boundary, So you take two copies of this and glue them together, and you get genus equal to $1 - n + \frac{1}{2}2(n-1) = 1 - n + n - 1 = 0$. So this is a sphere, and half of it is like the n-fold Vlaschke product. The Riemann surface is something where if you glue it to itself you get a sphere, so it must be a disc.

If you're a little bit uneasy about what I'm doing here, this part might be hard to follow, but I'm going to return to it again. I just wanted to point out that there are non-Galois covers, and, actually, think of this as an example, instead of.

So let's start a little topology now. Riemann called it analysis situs; we now call it topology. This should be called definition -1, since no one makes it anymore. These definitions will be over the completd z plane minus a finite set.

Analysis Situs Definition 1 *A multivalued function on $\bar{\mathbb{C}} - n$, with a nonremoved base point is a function on the paths in the domain beginning at the base point which gives the same value on two paths ending on the same point if the two paths are deformable to one another fixing the endpoints.*

So we've seen examples of this, our examples are these algebraic functions y which satisfy some equation $y^n + a_{n-1}(z)y^{n-1} + \ldots + a_0(z) = 0$. So these are mapping into this range number space also completed at infinity. Later they can have values anywhere, and we can talk about whether they're continuous.

you can think of this as a function on equivalence classes of paths, which is in fact what Poincaré did fifty years later. That's kind of done so that you can remember some results and forget what this rests on. The base point isn't so trivial. If you're in a static position it won't matter so much, but if you're moving it matters.

So you go along a path, extending power series around given points. Or you can think about doing it on sheets.

[Fire Alarm Rings]

Well, I could finish before they find us. People can leave if they want to. Oh, so y is one of these branches of a multivalued function, because if you're away from the branch point and you deform a path a little bit, you'll get your endpoint in the same sheet. So I've been using that continuously for the last two weeks. So this is an example.

Question: are the coefficients rational functions in z? They're polynomials so that they're defined everywhere.

Question: doesn't this depend on the value at the base point? Yes, but given a choice your function will be well defined on some neighborhood of the base point.

For these examples, the multivaluedness is finite. The definition would allow infinitely many. Riemann proved the converse.

We could immediately extend this definition to surfaces spread over the z-plane. We could define it anywhere that we have paths. Also we could talk about what lies over a region.

Analysis Situs Definition 2 *Part of a surface spread over the z plane is called simply connected if it is connected (i.e., there is a path connecting any two points, and any two paths with the same endpoints can be continuously deformed to one another fixing the endpoints.*

Remark 1 *So on a simply connected region a multivalued function will actually be a single valued function of the fixed endpoint of the path only.*

Example 1 *These are simply connected:*

- *the sphere*

- *anything bounded by arcs joined into a circle in the plane*

The annulus is not simply connected

The annulus was called by Riemann doubly connected.

Analysis Situs Definition 3 *A surface is called doubly connected if one cut makes it simply connected. A cut is apparently removing an embedding of an arc or a circle. Generalize to n-connected. The connectivity is one plus the number of cuts it takes to make the surface simply connected.*

Let's look at some examples. The twice punctured disc is 3-connected. The torus is also triply connected. Any cut leaving a connected surface leaves an annulus, which cuts to a disc.

Exercise 1 *What is the multiple connectedness of the surface with two holes? Draw the sequence of pictures.*

Well, so, what Riemann did with these ideas is he was studying, in general, he made a general theory of multivalued function. His functions were always differentiable in the sense of the complex variable by studying the discontinuities across the cuts of the single valued functions on the simply connected domain.

We'll return to this later. Somehow this topology was a powerful insight into studying these functions. So there's no class Friday, there's class Monday. So hand in these four homeworks to Scott on Monday.