I just wanted to say, I have Poincaré's collected works in my office, it's a stack about yay high, whereas Riemann's work is all in this book. But each one is a gem. I think of Poincaré as just a smart guy who extended what Riemann did, and Riemann is the one who made the big jumps.

Last time we saw how the fundamental theorem of algebra leads to a multivalued functions which are solutions of $F(z, y) = 0$ of the form $y^n + a_1(z)y^{n-1} + a_2(z)y^{n-2} + \ldots + a_n(z) = 0$. We'll assume that by moving the axes slightly we can separate out the variables like this. This is true except in degenerate cases. So when we evaluate at a particular z, we have at most n roots, and at least 1. Assume that at some value of z there are n roots. The roots will coalesce at most at finitely many values of z, so that outside of a finite set of zs (where the discriminant $(a_1(z), a_2, \ldots, a_n(z)) = 0$) we will have n roots.

As you start moving z around, these roots move around. Actually, that makes me think of Weierstrass. That makes me think of how power series change as you move their center. What Riemann did completely revolutionized what was going. He showed that certain problems could be associated with geometric pictures and conversely that every picture could be realized. So he completely finished it off, and at the same time Weierstrass came along and finished everything off with power series and analytic continuation.

Weierstrass noticed that Riemann's proof was a little suspect; the connection I made in the history was that there was a kind of competition going on, they were solving the same problems. Riemann's proofs were like magic; he just drew a picture and everything would become clear, kind of like Thurston.

So this object, the zero locus, is an abstract Riemann surface. In the nineteenth century these were imagined as spread out over the z plane and in sheets above and below it. This is sort of an element of the abstract definition of topology which is lost today. These are actual functions and they're main players in what's going on. But if you abstract away into abstract definitions of topology, a lot of the richness is lost. So I want to situate this before I distill into the topological forms.

We looked last time at $y^2 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)(z - \lambda_4)$. You can also add in the point at infinity, at $\hat{\mathbb{C}}$, which we see as the sphere, with antipodes 0 and ∞ and diameter the unit circle. So then the sheets are larger spheres around this one. We made slits on each sheet and glued together along them.

We studied what happened with only 1 root, where you get the standard square root picture. Then you get the regular square root picture, the ray. With the four factors the same kind of thing happens near any of the roots because the other factors are basically constant. Going around λ_i is just like going around the same point in $y^2 = (z - \lambda_i)$. So it was natural to put a slit between any two of them. So if you go around two of them the roots interchange and then interchange again. The best way to visualize this is to take the sphere inside the other and instead attach it on the outside. So glue together two twice-punctured spheres along separate punctures. So this gives the Riemann surface of a torus.

With three factors you also get a torus, pushing one of the punctures off to infinity. So you can write a little table

$roots$	$riemann surface$
$1, 2$	S^2
$3, 4$	T^2
$5, 6$	$T^2 \# T^2$
$\vdots$	$\vdots$

Riemann proved that any closed surface can be deformed to one of these. He also noticed the involution $y \to -y$; you have a number of fixed points equal to twice the genus.

Let's do y^3. I forgot to do that last time. When I was doing it myself I could see some of the pictures but I couldn't see all of them. This will basically be y^k, but let's do $y^3 = (z - \lambda_1)$ or equivalently $y^3 = (z - \lambda_1)(z - \lambda_2)$. So let's start with $y^3 = z$. The three roots of this complex number are symmetrically deployed around the origin.

We just did the case with one factor where λ was zero. If you go around λ in a picture with more roots, it will look the same around a small enough circle. So if we want to make a Riemann surface for this, we take a big disc and slit from each root to the edge. Take three copies of this surface. So the roots have the following structure. They're in a symmetrical arrangement, and you can multiply by a cube root of unity to get another root, so that you have an action, a symmetry. So the roots have this additional structure, and so we take three copies of this surface labeled by $\{1, 2, 3\}$, so that when we travel around in the right direction, the roots are subjected to this rotation, so that you go up like in a parking garage cyclically from 1 to 2 to 3 back to 1 as you keep going around. So it's a "Back to the Future" kind of parking lot.

I'll take one more slit on the other side, so a disc slit from the center (∞) to the boundary. I'll make them fit by looking at what happens as you go around the whole multiply slit disc. If the number of roots were divisible by 3, you'd get back to the identity so you'd just glue on three discs. But instead in this case, with five roots, you get something like the inverse, so that you want to glue the infinity discs to one another in the reverse order $1 \to 3 \to 2$. Just like, for y^2, you got a branch point at infinity if the number of roots didn't divide 2, here you get a branch point if the number doesn't divide three. If the number of roots divides the order of y, infinity plays no role.

Now I tried to work out what some of these surfaces look like, and I must admit I didn't get very far. So from here on out, for y^k you do the same thing, slitting from each root to a bounding disc and then gluing k copies of these together according to the order $1 \to 2 \to \cdots \to k \to 1$, and then slitting once from ∞ in another disc and gluing according to the number of the roots, i.e., through what happens as you go around the whole multiply slit disc. So what happens, what surface do you get? Well, the answer is seasonal. What does that mean? What season are we in? The season is fall. But culturally, if you drive around you see preparations for Hallowe'en. So these pieces look like the wedges of a pumpkin, and fit together something like that.

Exercise 1 *Make a picture for $y^3 = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)$. So there won't be anything at infinity.*

If you did it with two factors, then it would be the same, because there would be a factor at infinity. So take three thrice slit spheres and make the parking garage. It should have three boundary components, so you glue on three discs. It's very good to try to do these things.

So Riemann sort of worked out theories on this. His drawings just devastated the theories. The other mathematicians were afraid and jealous. They couldn't be mad at him too long, because he died. Another part of this story is that he had tuberculosis. At the age of 24 he created the theory of functions of one complex variable. His second thesis at the age of 25, was on Riemann surfaces and curvature. His thesis advisor was Gauss. Soon afterward he found out that he had was called consumption in those days, so between school semesters he had to go down to Italy, where he worked with Betti, who generalized what he did to higher dimensions. Then later Poincaré came along, and Lefschetz, and they modernized it all.

So some of the results they got were the genus of surfaces. Any closed surface can be deformed to one of these. So they called the number p, and we call it g. There's a minor thing here, which is that the surfaces should be orientable.

In any mathematical discussion you give some objects and then try to classify them up to isomorphism. Well, if you've done that, you're not done. You have the recognition problem. Given an object, which one of them in the classification is it?

This is one of the rare cases where you have a solution to both of the problems. Riemann solved this as well. There's a theorem, Riemann's recognition result. The formula is now called the Riemann Hurwitz formula because Hurwitz worked on it a lot. g is $1 - \#$ sheets $+1/2$(total defect). The defect is the number of extra roots that sit on top of a given root.

So for the exercise, there's no defect at infinity. Three roots have collapsed to one at each point so the defect is 2. So you get $2 + 2 + 2 = 6$ defects. Then the genus is $1 - 3 + 3 = 1$.

The formula is for all of these coverings, for any Riemann surface branched over the sphere. So this is the theory of the Euler characteristic. As soon as you get into sort of interesting examples, the pictures become quite challenging and you need this theory. He was very clear that these did not involve geometry. In his paper he called it analysis situs, which is also what Poincaré called it. Lefschetz called it algebraic topology.

In the general case you're going to have the z plane, enough of it to cover all of the finite branch points. At a point, suppose you have n distinct roots, so that you can take a big circle around the finitely many points where the discriminant is zero. So the picture is complicated. You cut out to the boundary from each point to keep from going around these bad points. Now any two curves that you draw can be deformed without a problem going around the branch points, because you can't go across the cuts.

So choose one of the roots as a starting point, and you can follow it to every place, so that by following the root along a path you can lift up to the next space up. You'll get a lot of disjoint copies, and they'll be glued together along these cuts, which is what I've been describing.

Some of you should have asked, why is this independent of how you do the cuts? Logically they're different if you do the cuts differently. But in the four dimensional complex space, different projections will give different pictures of the same locus $F(z, y) = 0$.

What were Riemann, Poincaré, and Betti doing? They were trying to solve systems of equations. But abstractly, to describe this locus we'll describe the multiply split finite part and then the singly split infinite part. Every permutation of a finite set is a disjoint union of cycles. You just group the sheets together according to the rules for each root. Abstractly these may be disconnected. The constant 1 is probably the number of components.

So Riemann has arrived at abstract surfaces. We can look at the equation, read off its shape from the formula. So this is a deep invariant, the genus of an equation. And it can be extended to the degenerate cases.

Let's just recall a little bit. Write $y^2 = x^2 + 1$. So what's the genus of that equation? If you break the right hand into two factors you get two sheets and two defects so the genus is $1 - 2 + 1$. For $y^2 = z^3 + z + 1$ you get $1 - 2 + 2 = 1$.

In the case of the circle, you can substitute, say, t^2 at 1 for $t = 1$. Then subsitute $y = u^2 - v^2, z = 2uv, t = u^2 + v^2$ and you get $(y/z)^2 + 1 = (t/z)^2$. This gives

$$\frac{u^2 - v^2}{2uv} = \frac{1}{2}\frac{u}{v} + \frac{1}{2}\frac{v}{u}.$$

So this is called a rational parametrization, or a rational curve. Weierstrass looked at the parameterization for genus 1 surfaces; these are elliptic curves. People are still looking at higher genuses. You can also study the rational solutions to these equations. For example, look at $y^n + z^n = 1$. This is the Fermat theorem. So it does have some rational solutions. Fermat's theorem is that these obvious solutions are the only solutions.

But earlier than the recent work on Fermat was the Mordell Conjecture, that an equation $F(y, z)$ of genus greater than 1 with rational coefficients has only finitely mny rational solutions. This conjecture is also made in any finite extension of $\mathbb{Q}$. This is not true for elliptic curves. The group structure preserves the set of rational points, and that set turns out to be a finitely generated abelian group which sometimes has nonzero rank.

The conjecture was solved by Faltings in the 80s. What I like is how this question is sort of pinned down by what Riemann was doing.

Let's calculate the genus of the Fermat curve, using the formula from before. It's $1 - n + (n-1)(n)/2 = (n-1)(n/2 - 1) = (n-1)(n-2)/2$. So we get

genus of n	*Fermat curve genus*
2	0
3	1
4	3
⋮	⋮

So I felt like the proof of the Mordell conjecture was basically the proof of Fermat.

So why don’t we say homework is due the Monday after next.