Algebraic Topology
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So I'm in the midst of proving the workhorse result of Riemann that would let us say that those are all the ways, well, anyone have any homework? Are you still thinking about the roots moving continuously?

So, um, we're trying to show a closed orientable combinatorial surface is determined by its Euler characteristic. So we've started from a general combinatorial surface and went to having one face glued along some graph. Then we put a hole in it and took out an annulus, so we were looking at a neighborhood of a graph. I'm putting twists in because I want the boundary to be connected.

And then we took a maximal tree inside this graph and then we had this neighborhood, and inside the neighborhood of the graph you take a neighborhood of the maximal tree and then see the rest of the neighborhood as attaching bands. Then this becomes a disk with 1-handles attached. They have to be orientable and have only one boundary component.

So now we want to analyze this, the disk with handles. The twists are only in the drawing, it's because we want only one boundary component. We'll see in a minute why we want an even number of handles. See we've got an even number of points on the boundary and then some pairing given by the connections of the handles. Now if you have $2k$ points there are $2k - 1 \cdots 7 \cdot 5 \cdot 3$ ways of doing this. There's only supposed to be one way to do this up to equivalence. This will be similar to Riemann's cuts.

So start with our disk and put a handle. Up to equivalence there is one way to put a handle on. I'm going to put the handles on one by one, but I'll be smart about it. Now this has two boundary components. If all of the rest of the handles were on one side or the other, the boundary would not be connected. Then there is a handle connecting the two components. So I can slide them around since we're working up to equivalence, and if I put one in connecting the components I get a disk with two handles and one boundary component. If I add another handle again it's canonical and forces the same sort of choice. So what is this guy with two handles and one boundary component? It's the torus with a hole.

So now I attach another handle and get a torus wearing a pair of pants, the twice punctured two torus, and then another to get the punctured two torus.

So it just continues this way. You put a handle on, and then close it up, and so to get to a connected boundary, after doing this step k times, you've got $2k$-handles and a connected boundary. So the picture from the graph is equivalent to the construction of the n-torus.

I'm thinking of this equivalence as smooth homeomorphism, combinatorial equivalence, diffeomorphism, you can have a self-contained discussion of combinatorial equivalence.

So for a Riemann cutting scheme everything is similar, reading it backward, except you start from a closed surface and you have marked points. There's kind of a shift by one, since in this case we've got a surface minus a disk, and so it's the same number as the number of cuts. The disk we started with was the $8k-4$-gon. Now, $1 - \#handles = \chi - 1$, so since the number of handles is even you can write it as $2g$. Then we let g be the genus, and it's defined by the Euler characteristic. Now it's the number of handles, or half the number of cuts in Riemann's process.

So, any questions about this argument? Start from a definition of a cell complex locally like a surface, and then just work with it combinatorially and pictorially.

[Scott: can you think of these procedures as taking place in moduli space?]

That wasn't the question I thought you were going to ask. I won't entertain that question officially because the word moduli space was used.

[Scott: Will you answer the question you thought I was going to ask? No?]

Yeah, yeah, I'm going to draw a picture. Well, lemme draw a variant of this. This is about getting down to one, well, let me just draw different pictures. Suppose you have this cell decomposition. I'm not going to worry about having only one face. We used the idea of handles here. We got a handle on this by using handles. So the picture I want to draw is of my cell complex, and around each vertex you put something. This is something you want to do anyway. You want to use the face that these things fit together nicely. So you want to draw a disk to make things look like the Euclidean plane.

Then you can make a road between each of these, by widening the edges of the 1-skeleton. Then you get 1-handles attached to the disks.

Now we have a completely symmetric picture. We have a bunch of disks with 1-handles, and these things like faces, you could put holes in them and retract them down. We can sort of do that as well. Now we could do lots of things with this picture. We could say we don't want to have so many disks, we want only one disk. So we amalgamate along a tree of disks and get back to the same picture. But you can also consider the dual picture whose vertices are faces and whose edges are crossings of edges from face to face. So you can actually have two pictures of a single disk with handles or many disks with handles. So adding a handle to one side is like making a cut to the other side. So there's a sense in which adding handles on one side is the same as making Riemann cuts on the other side. The Euler characteristics are the same since you're just switching F and V. But these are actually homeomorphic things, just in general position. Now they may have some differences, but this is true once you've gotten down to one disk. This is called Poincaré Duality. The proof is always the same. So the two processes we're talkinga about are in duality, adding handles and making cuts.

So we showed our desired result, and also showed that $2g$ is equal to the number of 1-handles, or number of Riemann cuts. We haven't quite shown minimality.

Thirdly, an orientable surface with boundary is determined by the number of boundary components and the genus of the associated closed surface, obtained by gluing a disk to each boundary component. If you take two surfaces with boundary with the same number of boundary components and equivalence associated closed surfaces.

So the surface minus disks is equivalent to the surface minus a regularized little disk. You can think of drawing these as a long string of buttons on the surface. Then it doesn't matter which ones you're filling. This step works in any dimension.

[Scott: What does that mean?]

Removing k balls only depends on k.

As another corollary, when we're doing Riemann cuts, say on a surface of genus g, is that cuts are equivalent; you get the same thing no matter which cut you choose. The surfaces you get have the same Euler characteristic and the same number of boundary components. Then there's an equivalence from any one to the other.

So you have to do rotations to get your boundary components to match up. This is worth thinking about. Anyway, there's only one way to do the first cut, then you do the second cut and get a square. So again you have to rotate so that you're matching up appropriate sides, and then you have to stretch along the intervals. That shows that there is a unique closed curve which doesn't separate a surface with two boundary components up to homeomorphism.

This is not true for systems of two closed nonseparating curves in a closed surface.

There's actually something called the arc complex where you make a structure out of all these curves.

Now I have to worry about the corners, so the next cut is not unique. The argument will be that when you cut open, the object will be determined by this piece of information, which gives you the number of sides to the boundary components. So this number will determine the structure that you get. So each cut is unique if I ignore the other combinatorial structure. But if I put them together, well, I kind of got ahead of myself there. We want to say that Riemann cuttings are unique, ignoring memory.

If I look at the proof of the handle system backward, everything is equivalent. So any Riemann cutting system will look like cutting the handles backward, if you just think of the stages. If you look at them altogether then there are quadrifactorial ways. You have to remember the n-gons.

I see this intuitively, but I still want to see formal proofs. Let's see the three for genus two.

Exercise 1 *Show that closed nonorientable combinatorial surfaces are determined by their Euler characteristic. Now you'll have twists to contend with. That is, get to a disk with* 1*-handles and then look at it anew.*

What are the invariants?

Let me discuss something a little bit. Suppose I have a closed surface, orientable or not. Take a closed curve γ and look at a neighborhood of it. The neighborhood is either equivalent to an annulus, in which case you say $\epsilon(\gamma) = 0$. You say it is 1 if the neighborhood is a m obius strip. It's the number of boundary components modulo two. This function from closed curves to integers modulo two, we'll call w_1. What equivalences preserve this map? It's true up to isotopy because the neighborhood doesn't change up to homeomorphism.

First of all this is defined on isotopy classes. We can extend this to unions of closed curves just by adding. We add up the number of boundary components modulo two. So two strips give $1 + 1 = 0$. There's an operation you can perform which looks a little like string topology. You can replace a left-right division with an up-down division, i.e., look to the opposite resolution of a crossing. So this works even through intersection. So now you can define it for any union of curves.

So the claim is that if you take all closed curves, unions of them, modulo isotopy and the resolution relation, then this is a finite dimensional vector space over $\mathbb{Z}_2$ of dimension $2g$ in the orientable case. This will be called the first homology group with $\mathbb{Z}_2$ coefficients. This w_1 is defined on it, and measures whether or not a cycle is twisted or not.

There is an oriented analogue over $\mathbb{Z}$, but here you have a preferred smoothing.