Algebraic Topology
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So did anyone do the problem about other invariants of subdivision dependent only on V, E, and F? There is one, and it's dimension, and the two of them classify complexes up to subdivision. So if you have X_1 and X_2, and $\chi(X_1) = \chi(X_2)$ with $Dim(X_1) = Dim(X_2)$ then $\phi(X_1) = \phi(X_2)$ for any function $\phi(V, E, F)$.

Proposition 1 *If X_1 and X_2 are cell complexes with the same dimension and the same Euler characteristic then there are subdivisions of the two complexes with the same numbers of faces, edges, and vertices.*

If two complexes have the same number of vertices, good; if not add vertices by subdivision to get to complexes with the same number of vertices. You can do this in any dimension below the top (the top dimensions are the same since they share dimension). Then you know that all the numbers of complexes up to $dim(X) - 1$ are the same, and with the Euler characteristic that gives you that the last number of cells, of the top dimension, are the same.

If you didn't have cells of the top dimension, you couldn't do this. You have to have these two bits of information. When I was thinking about the question, the hardest thing was to come up with the statement. Once you have the statement, it's relatively easy to prove.

These two invariants are probably the most important invariants. They're simple and they occur all the time. I mean, they're not very deep, but they're deep in some sense. Poincaré and Riemann spent time discussing what dimension means quite a bit.

Okay, so now I want to discuss the, well, are there any questions about the other exercises?

[Scott asks for a definition of when two cuttings are equivalent]

That's sort of related to what I'm going to be talking about now. Well, uh, equivalence requires that a homeomorphism of the surface takes the one cut to the other cut. Any two cuts on surfaces of the same genus are equivalent. This is beacause of this:

Theorem 1 *Two orientable surfaces of the same genus and same number of boundary com-*

ponents are equivalent (homeomorphic or diffeomorphic).

You also need a lemma, or fact, that the group of orientation preserving diffeomorphisms of the circle is connected, well, path-connected.

To do this you show that any orientation preserving diffeomorphism is path-connected to the identity. You can pull back to the identity in a clockwise fashion. You have to sort of make sure that the map is smooth.

Anyway, if you have two boundary components with the same number of sides, the same argument applies.

So you have two gluing maps, that give you smooth bijective equivalence between these two surfaces with boundary.

In the first step you can use an isotopy to the identity over a small collar near one of the holes. It starts at the identity and goes up to something else. Then when you look at the diagram locally you get the identity. The obstruction is that I need to get the two boundaries to match up. I want the gluing up to be the identity. So I change it a little locally and am able to glue things up.

As we go through the cuts, the notion of equivalence is becoming a little more stringent, in that the points are now part of the structure. By the previous argument, if you ignore the points, any two such are equivalent. So two cuts with the same number of points on each side are equivalent. That's a discrete invariant of cuts, since there are a finite number of cuts. The way I phrased it was to show that there are at least this many. This is how you show there are exactly this many.

So let's prove a variant of this statement. Let me put "pre"-theorem on the result above. But we haven't really defined a surface.

Definition 1 *A surface is either one of Riemann's surfaces spread over the z-plane and completed or it is a $2D$-cell complex satisfying two properties:*

1. *Each edge is on the "boundary" of one or two faces. If it's a closed surface, make that two.*
2. *At each vertex we have one of two structures; either there are a cycle of faces around it or there is a sequence of faces around it.*

The second definition subsumes up to equivalence the first definition.

If you take a smooth surface with boundary and break it up into a cell complex you get something that looks like this. You have to say that it's compact, it's Hausdorff, if it's not compact it's paracompact, and so on. You have to start talking about atlases.

It turns out that everything you do in analysis does these automatically.

There was a long time when people thought they were understanding Riemann by doing this. I have a history of it.

For my money this was negative progress, because they abstracted everything and people forgot the original problems. But that's my personal opinion and you can pretend that I didn't say that.

You can do this in n dimensions by a subtle refinement. Brouwer came up with things that only looked like rectilinear subdivisions of Euclidean space after subdivision. These are called combinatorial manifolds.

Up to dimension six these have unique differentiable structures.

At seven, Milnor's thing he did when he was 57 or something showed that the different manifolds, the same manifold could have different differentiable structures. In dimension 10 there's a combinatorial manifold with no differentiable structure.

The good news is that the number of differentiable structures on one of these is finite. You get them using cohomology, which we're going to define.

That finite difference is still being discussed intensively. I was just at a conference and almost every talk was about this.

What about topological manifolds? Everything works kind of nicely with these cell complexes. If you have two cell complexes which are homeomorphic can you subdivide them to get a common subdivision?

Again, Milnor found the counterexample, in dimension five or six. They knew that the homology was invariant under subdivision and wanted to show it was a topological invariant. If two homeomorphic things have isomorphic subdivision that shows that it's a topological invariant. Brouwer formulated this.

Alexander, from Princeton, who only really dabbled in math and then became a banker, I mentioned this in the blurb about the course, you can approximate any map with a map from cells to cells. This gave a very easy proof that this was not just an invariant under homeomorphism but under homotopy. So this cruder category arose.

I studied this for manifolds. You can almost prove it for manifolds of dimension bigger than four.

Another big development, in the eighties, were when Donaldson found an infinite set of invariants in dimension four, which are combinatorial invariants but not topological invariants. He presented them as smooth invariants, because this is true up to dimension six. There are infinitely many four dimensional manifolds, all of them homeomorphic, but they're different as combinatorial manifolds or smooth manifolds.

So that's eighty years. They're still kind of new, because we don't understand them well.

A very intriguing problem is to find a truly combinatorial explanation of these, like an Euler characteristic.

But there are other problems of the same ilk which are easier and unsolved so maybe you shouldn't look too hard at this.

So now to classify into orientable, we assign to every face a direction and then we want things to cancel on adjacent faces.

You probably know that the mobius strip, when you try to do this, they disagree. So it's not orientable. And I could put, um, and remember, the definition of genus for a closed surface, and surface is now defined combinatorially, genus is defined by the Euler characteristic $=$ $\chi = 2 - 2g$. The theory of cuts depends on this so for a logical definition this is what we'll take.

The Euler characteristic is invariant under subdivision, which we sketched. Now once we show that the Riemann system is well-defined, we'll be able to use the old definiton. We're in the midst of that right now.

The genus of a surface with boundary is defined by genus of the associated closed surface. The associated closed surface comes from putting a disc on every boundary. So then we can make a formula. The Euler characteristic of the closed surface is equal to the the Euler characteristic of the bounded manifold plus the number of boundary components.

Recall that $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

So the genus is $1 - \frac{\chi + \#\delta}{2}$. Did I do that right? Yeah, so this is for orientable again.

[Nate: could you say again what you mean by associated closed surface?]

You glue a disk on every boundary component. So every surface has an associated closed surface, and this gives you the genus.

So these have to be orientable. You'll have to modify it if you want things to be nonorientable. So let's start proving this.

Let's do the case without boundary. Then I'm going to reduce to the boundary. I'm doing the case without boundary but I'm going to use a lot of surfaces with boundary.

So around every vertex it has to be cyclic. The first step is to get one face. You start somewhere, and, you know, some of you, you get a place, break down walls and get bigger rooms. So you unsubdivide, and either there's a room you haven't got to, so you break down the wall, or you're done. On the interior of this thing you're just combining n-gons along a face and erasing the face.

So you're finally left with one two-cell. So you have a graph of the things that are left and a big n-gon with this thing sewed on to make the surface. If you have little prongs sticking out you can get rid of them too.

So now you get a trivalent graph (generically). Since this is a surface the sewing passes twice, once on each side. Now let's punch a hole in the top cell. What happens when you poke a hole in a balloon?

[Everyone: it pops]

Well, all the air goes out

[You never said it had air in it.]

The only true mathematician, no common sense. So you get just one curve, because it's only got one boundary component. You can put twists in so it only has one. So we want to try to understand that thing.

So there's another step, where you want to take a maximal tree, which is a subgraph of the graph, which is a subtree inside the graph which contains all the vertices. There are many of them. You can always do that. Oh, why do maximal trees exist? This graph is connected because it's the image of a circle. so any two are connected by a path. Start with a vertex, and keep adding things with paths back to where you started. By some induction a thickened surface around a tree is a disk. So what we can do is collapse this maximal tree to a point. Then you have a big point with some edges, those which weren't in the maximal tree. So you think of the disk, with other edges with things attached.

I'm supposed to end up with one boundary component so you have to alternate their starting points. So we see our surface as a disk with one-handles. Our original surface is homeomorphic, diffeomorphic, to a disk with one-handles.

This is not a very formal proof, but pictorially it's something that you could see is correct, in a cell complex.