# Algebraic Topology <br> October 29,2004 

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Every nonorientable closed surface is combinatorially equivalent to a disk sewn to the boundary of another with $h$ nonnested nonintersecting twisted bands.
[Interlude where I prove it]
If you just have one twist, it's the real projective plane. If you have two collections with $n, m$ twists, you can look at their connect sum as having $n+m$ twists pretty easily. So $\Sigma_{h}=$ $\#_{h} \mathbb{R P}^{2}$. You can also look at it as $\mathbb{R P}^{2} \# T^{2} \# T^{2} \cdots$ or $\mathbb{R P}^{2} \# \mathbb{R P}^{2} \# T^{2} \# T^{2} \cdots$, depending on the parity of the genus.
$\mathbb{R} \mathbb{P}^{2} \# \mathbb{R} \mathbb{P}^{2}$ is the Klein bottle, which you can see because the torus is a double cover of it or just directly geometrically.

If you look at the algebra of nonorientable surfaces, it is just $\left\langle T^{\# n}\right\rangle$ If you throw in orientable ones, it's $\left\langle\left(\mathbb{R}^{2}\right)^{\# n}\right\rangle$ but if you want all of them, you need the relation $T \# \mathbb{R} \mathbb{P}^{\not \vDash}=\mathbb{R} \mathbb{P}^{\not \vDash \# 3}$.

Does this process remind you of any algebraic process? It's kind of like putting a basis in standard form.

So I wanted to summarize a little bit what I said last time and then go forward. There are two paths to go forward, and we have to choose one, well, I've chosen one. We talked about classical physics where you had the gradient, curl, and divergence. Then we had that this was equivalent to something with differential forms


So differential forms were the natural integrands. The reference here is to Spivak's Calculus on Manifolds. Then Stokes' theorem said that $\int_{\delta R} \eta=\int_{R} d \eta$.

Next, we looked on a closed orientable surface at a harmonic form, locally $d$ of a harmonic
function. I didn't develop any of the theory of this, but we showed that a harmonic 1-form is determined by $2 g$ periods. So to write $\eta$ as $d f$ you get a multivalued function for $f$. But because of Stokes' theorem, $f$ was abelian; if you deform or homologize the path then you get the same thing. The harmonic forms are at most of dimension $2 g$; I forgot that I didn't prove the existence of any of these.

Riemann cut these up into simply connected things and then glued them together and got things to match up; this is known as solving Dirichlet's problem. It was in dispute for a while because the argument used physical principles that don't work in all settings. So we have half of the Riemann result that on a Riemann surface, there's a multivalued map to $\mathbb{C}^{g}$, where you get a picture which turns out to be a lattice. This is another reason it's called Abelian, because the lattice is an Abelian group.

In general you get a base lattice and divide by it to get $T^{g}$, a complex torus of dimension $g$. It's a well-defined map of the Riemann surface into the complex torus. It's a highly studied object. You can think of the lattice as $\mathbb{Z}^{2 g}$. I think this curve is an embedding except in what's called the hyperelliptic case.

We started with an algebraic surface $F(x, y)$ and then there might be an algebraic meaning to the analytic map from $\Sigma \rightarrow T^{g}$. This is like going in an algebraic geometry and arithmetic direction, but we won't go there. Abel had a role in this, but I don't know what.

This example is still very significant from an analytic, algebraic-geometric, arithmetical point of view.

So we could think of this discussion two ways; either it's the discussion of multivalued functions, or it's the discussion of homology. Since it's Abelian, it's related to the first homology group. We could go off into higher dimensional (abelian) homology, or into nonabelian multivalued functions. I'm going to go a little while in the direction of homology and then a little while in the direction of the nonabelian functions. These latter are linear representations of the fundamental group. They can be generated nicely using something called flat connections. This is sort of one-dimensional.

In some sense you could say it doesn't have a good higher dimensional analogue. In some sense it's Gauge theory, which studies all connections, not just the flat ones. But that's not even understood. But let's say something about the other thing we were talking about.

If we didn't want to mention holomorphic, we could have talked about closed one forms and mapped into $\mathbb{R}^{2 g}$. We get a $2 g$ period associated to a form and if it's zero we find a function. It turns out to be onto but we haven't shown that. Say we've shown it's onto; then closed one forms modulo exact one forms is isomorphic to $\left.\mathbb{R}^{2 g}=H_{1}^{\text {dual }}\right)_{\mathbb{R}}$.

Let me get an exercise on the board here. Let's go back to the plane again. No, three dimensional space. When Betti interacted with Riemann and generalized things, well, let's look at a region in $3 D$ space, like take out a knot or some balls or lines, and then there are closed forms like $\omega=\frac{x d y d z+y d z d x+z d x d y}{x^{2}+y^{2}+z^{2}}{ }^{3 / 2}$. This is defined off the origin.

Exercise 1 Show that $d \omega=0$. We can form the integral $\int_{S} \omega$.

If we move the surface by a homology the integral is not changed.

Theorem 1 If you have a region $R$ in $3 D$, and $\omega$ is closed in $R$, and you want to know if you can write $\omega$ as $d \eta$, you can do this if and only if the integral of $\omega$ over every closed surface in $R$ is zero.

This was known to Betti in the 19th century. So we say $\omega_{1} \sim \omega_{2}$ if $\omega_{1}-\omega_{2}=d \eta$. The number of linearly independent closed integrands is something like the rank of the second homology group, which I've defined. Because of the theorem, it's at most that rank; if you look deeper it's equal to the rank.

The one in dimension two is $\frac{x d y-y d x}{x^{2}+y^{2}}$ is closed, and you can find the formula for $D=4$.
Exercise 2 Show these are closed and find the analogous expression for $D=4$.

