# Algebraic Topology
# October 22,2004

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December 13, 2004

I don't like jokes in math, it's so damn hard and serious. I should lighten up a bit. If you're not religious,...

So let's look at these differential forms in dimension two. You have 0-forms, and then 1-forms and 2-forms. You have the maps $d$ which take $f$ to $f_x dx + f_y dy$ and $X dx_Y dY$ to $(Y_x - X_y) dxdy$. So the composition takes $f$ to $(f_{yx} - f_{xy}) dxdy$, and if your functions are sufficiently smooth you can take derivatives in any order, so that $d^2 = 0$.

In all dimensions $n$ you have $dx_1, \ldots, dx_n$ as the generators for an antisymmetric algebra over a function space. Then $d$ acts on $X \prod dx_{i_j} = \sum X_{x_k} dx_k (\prod dx_{i_j})$. If you have a set $I$ with $k$ indices then a term in $d$'s image is in a set $J$ which consists of the union of $I$ with some $x_j$ not in $I$.

If you do $d$ twice and sum up the terms, you can add the two indices in any order. So this is $X_{x_i x_{i'}} (dx_i dx_{i'} - dx_{i'} dx)(\prod dx) = 0$. You get a lot of terms but they come in pairs and because of anticommutativity they cancel. This was the Poincaré result. This argument is worth thinking about because, well, I'm going to give this as an exercise.

**Exercise 1** *Write out a proof that* $d \circ d = 0$.

Many times in research I have to prove that $d^2 = 0$. Terms always cancel in pairs because they appear in opposite order. I'll show you another example. How do you know that the boundary of a boundary is zero?

Suppose, instead of a picture, we made things more algebraic? What if we did this in domains of integration of forms? So $R$ is $k$-dimensional and has an orientation (so that you can integrate over it). At every point you evaluate your form on a volume element, and you get a function, and then the volume element has a measure in it, and you integrate this out. But you have to have an orientation to do the integral, and when you write Stoke's theorem you get $\int_{\delta R} \phi = \int_R d\phi$. So you need to be orientable so that you can make a choice about induced orientations. You choose something like $(n^o)(surface\ orientation) = (intn)$. If you chop up $R$ into a bunch of regions you get that it is the sum of domains, and then you want to show that the boundary of the boundary is zero in these terms. When you look at the formula, you get reversals, a different sign. $n_2^o = -n_1^o$. So this is exactly the picture that you use for the form case.

This is one of the fifteen arguments that come up in mathematics, so it's worth doing it.

[I have a historical question. When did people notice that these things were dual?]

I don't know. I would have thought that they'd have this thought at the beginning.

Let me think of one more example of this. There's some interesting construction, where you take the characteristic zero vector space over isomorphism classes of oriented connected graphs with $k$ edges. If you have a graph, an orientation is a linear ordering of the edges up to an even permutation. There's a map $\delta : G_k \to G_{k-1}$. If you take a bunch of real variables you need an orientation. You need $(\Gamma, ) = -(\Gamma, -)$.

This is not to be confused with a directed graph. Now there's a map where you take an oriented graph $\Gamma$ and map it to the sum over the true edges (not loops) of $\Gamma/e$, where you collapse the edge to a point. Orient modulo $e$, defining this implicitly so that $(e, /e) = $ .

It's better to say that you move the edge to the beginning of the orientation, adding a sign, and then eliminate it.

This is a boundary map

$$\cdots \to^{\delta} G_k \to^{\delta} G_{k-1} \to^{\delta} G_{k-2} \to \cdots$$

**Lemma 1** *(Kontsevich)* $\delta \circ \delta = 0$.

So you have a tree of collapsings, and after you mod out by isomorphism classes you get coefficients. If you collapse $e'$ and $e''$ in one term then you can collapse them in the opposite order in the other term. So then you get a negative sign and they cancel. The subtle point is that maybe $e'$ and $e''$ collapsed in the other direction create a loop in a bad way. But then this element is the zero element; it's isomorphic to itself with the opposite orientation.

So there's something called graph homology, and it's defined by this.

If you shake a bottle of beer and look at the soap suds, this kind of looks like this collapsing of edges. Next time you have a beer, just look in there.

[Is there not a differential on isomorphism classes of higher dimensional $CW$-classes?]

I'll think about it. Oh, there won't be a class on Monday. I'm going to give you a 4-D exercise.

So $d^2 = 0$ in the 2-manifold case exactly demonstrates this argument. Say $\gamma = \delta R$. Then Stoke's theorem says that $\int_\gamma \omega = \int_R d\omega$

Actually, it's an interesting question. When can you define the interior? You need a certain amount of regularity. Let $xdy$ be a 1-form. Pictorially, integrating on the boundary gives an integration over the region, as Stoke's theorem says. It's kind of a picture of a proof. So this has a practical example, which is the tool that architects or surveyors use to calculate area by tracing a boundary.

So the forms and these algebraic rules are all coming from the idea that these forms are the natural integrands for multidimensional integrals. When you define integration and then you change variables, the sizes are altered by derivatives. So you can see in the limit how to change variables. The change in area is given by the determinant of the matrix of the partial derivatives.

What this means is that we can set up forms on an abstract surface. If it happened to be spread in sheets over the plane, then away from the branch points you could just move on the plane, and over the branch points I even told you how to do it. In an abstract surface you have charts and you can change coordinates, so that a differential form comes coupled with a chart. A form has to respect the gluing map in this way. An abstract differential form will assign a form in a sensical way.

Then if you don't want to have an abstract surface you can look at the plane without some points. Then the idea is to consider functions, 1-forms, and 2-forms. What Riemann studied was 1-forms $\eta$, like $X dx + Y dy$, with $d\eta = 0$. Then the problem is, we ask, is there a function $f$ so that $df = \eta$? As we will see, it is not sufficient.

However, it is sufficient locally. You want to find $f_\alpha$ so that $df_\alpha = \eta$. You want to find $f$ up to a constant. So at a given point let $f = 0$, and take a path in the region, defining $f(x) = \int_\gamma \eta$. Then if your neighborhood is simply connected then $\gamma_1 - \gamma_2$ defines a region and Stoke's theorem tells us that the integral is 0.

You can try to do this globally, but if you have holes then choosing different paths may be problematic because you don't have Stoke's theorem. Then you get an ambiguity in $f$ by the integral of $\eta$ around a closed path. So I really have a stack of definitions.

[So integration is multivalued?]

Yes, you get a star! We have a star system at our house. The nine year old likes to get the stars, and the two year old wears them.

That's a good point, $f$ is a multivalued function. We can deform paths fixing the endpoints, but we can actually do more, as this is what is called an Abelian function. I'll discuss the property which these words correspond to.

[It's actually single-valued on the universal cover.]

Ssh! We don't know what is the universal cover. That's not the important part, the important part is to remember how it's constructed, which is from the paths. The name isn't important unless you want to communicate with the outside world. For me that's not important, it's second order.

For an abelian multivalued function, if two paths differ by a homology then the function takes the same value on them. So you can do the local moves of a cobordism. That's what we have here because of Stokes' theorem.

So for the surface of genus two there were four $\mathbb{Z}$'s. So the indeterminacy in finding $f$ such that $df = \eta$ is generated by the $2g$ periods of $\eta$. Then the difference between any two paths is equal to a linear combination of those four curves.

So you have a PDE constructed by a finite number of constants, which come from the homology group.

This'll be a little bit, Riemann was actually interested in special functions, Riemann surfaces, so his coordinate systems were related by diffeomorphisms of one complex variable. So we can think of ourself in a region of the complex plane. Then there's a notion of harmonic one-form. There's a notion of a harmonic function, where you could say $\Delta f = 0$ or that its value is equal to its average around a circle. A holomorphic function has as its real and imaginary parts harmonic functions whose gradients are orthogonal and of the same magnitude at any point.

A harmonic one-form is something which is locally $df$ where $f$ is harmonic. In this case, there's a wonderful fact only true in dimension two, where if you rescale the metric it doesn't change.

Let me just round this off. If $\eta = df$ then you've determined it up to a constant. Going back to the genus two surface, as a corollary,

**Corollary 1** *The space of harmonic one-forms is finite dimensional of dimension* $2g$.

If these patch together, and are defined compactly, then this is constant. So there are $2g$ parameters corresponding to the homology.

A one-form is holomorphic if it is locally $df$ for $f$ holomorphic, then

**Corollary 2** *The space of holomorphic* 1*-forms is a* $g$ *dimensional vector space.*

A holomorphic one-form is of form $\eta_u + i\eta_v$. One of these determines the other, and they are determined by $g$ real constants.

This was one of the great theorems of math. You have a Riemann surface, an analytic object, and then you use abelian holomorphic multivalued functions and form a $g$-dimensional space, where $g$ is now a topological invariant.

This interesting object only depends on the number of handles; this is like the crown jewel theorem here. What are the ideas? You just say words now, you generate topology. All of the ideas are here.

Next week then.

**Exercise 2** *In a way, this is a problem for me, too.*

*You have $0 \to 1 \to 2 \to 3$ in three dimensions. You had this nice picture, functions to vector fields to vector fields to functions.*

*Write down the same thing in dimension four: $0 \to 1 \to 2 \to 3 \to 4$. Verify that $0$ and $4$ forms are like functions, and $1$ and $3$ forms are like vector fields. What are 2-forms?*

*At each point, a vector field has four parameters. A two-form has six components, $\binom{4}{2}$. You have a six dimensional space. But now there's some natural structure. If you have a 2-form $\omega$ you can go to $\omega_1 \wedge \omega_2$, and get a 4-form. This is symmetric, so it's a quadratic form, and its zero locus is a cone. This has a signature, i.e., it could be $++++++$, $+++++-$, $++++--$, or so on.*

*So analyze these at a point. This is the last problem of this kind that you can do. You can't analyze the other ones. That's the exercise that I can do. What I can't do, is, turn the lecture around. You can turn physics into this kind of abstract math picture, but is there a physics interpretation of $d$ for $1 \to 2$ or $2 \to 3$. I think there's something to say because this is the last time you can analyze the whole thing, in four dimensions. You can sort of do five in the same way. The dimension of these spaces of forms, say you have $\binom{n}{3}$, this is like $n^3$. So you describe up to choosing coordinates, so even after modding out by $n^2$ for the basis you have parameters left. So the set of equivalences have moduli. There are a couple more dimensions where $n^2$ beats this number, but once $\binom{n}{3}$ beats it you can't figure it out any more.*

*If you want to cheat, go talk to Claude LeBrun. He lives in this picture. He writes it down every morning. The exercise has two parts. The part you can do is analyze the multiplication; the other part is to find the physics interpretation.*

*The quadratic form gives us some structure. The ones which don't have $\omega \wedge \omega = 0$ are called symplectic forms. There's a duality here. So the Einstein equations can be expressed in terms of this form. That's what Claude LeBrun does all the time.*