Algebraic Topology
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I want to give Poincaré's definition of homology and why he, well, I won't get to this today, but his motivation [Scott: fame and fortune] and 19th century background, well, what I know of it.

Basically, Poincaré knew about integrals over k-dimensional domains, and there is an elegant calculus over this, and if you dualize it you get homology.

This calculus had origins in classical physics. People like Maxwell and Faraday had pictures of electric fields and so on, which are good to know and give us a geometric picture of homology and topology and so on.

Let's work in ordinary three dimensional space, and then sometimes the plane as a special case, something repeated in the plane at every level.

The cast are functions and vector fields. Most plays are about murder or adultery or something, so you have some operations, for example the divergence of a vector field is a function. The gradient of a function is a vector field. The curl of a vector field is a vector field.

So we have, and actually there's one other thing, which is the cross product of vector fields, which is only true in three dimensions. Of course we also have the inner product of vector fields.

Is everybody totally familiar with these? If you compose two of these, sometimes you get zero and sometimes you don't. So now what are these thing, they have physical interpretation in terms of certain integrals. For example, looking first at the gradient.

In this situation the function in the equation $V = grad\ \varphi$ then φ was called the potential function for V, and V was called the force field associated with φ. Here the gradient is $V = (\varphi_1, \varphi_2, \varphi_3)$.

Then $\int_\gamma V\cdot$ (unit tangent to γ)$= \varphi(\gamma(1)) - \varphi(\gamma(0)) = \Delta\varphi$. So this is like the fundamental theorem of calculus in three variables.

The divergence of a vector field is a function. Up to a constant the function is determined by the gradient. So, first let me say something about the geometry of a vector field. What you can do with a vector field, if you have a vector field, you can solve, with a differential equation, get a little local flow, a motion of space. By the way, this is an aspect that is not going to be part of the homological motivation.

This is a higher order consideration but I think it's good to have this in mind. But you can think that a small region of space moves to another region of space. If you let time go to zero then you get a linear transformation, the infinitessimal perturbation of space. S this is $\lim_{t\to 0}(D_t - Id)/t$, which is an endomorphism of the tangent space.

Now, every matrix, if you take a small sphere, if you apply a matrix to it you get an ellipsoid. You can read off the volume distortion, the shape distortion, and the rotation. Every matrix is the composition of these resizings and rotations. The rotation is given by a skew symmetric matrix. This is the curl, and the German word for it is rot. The volume distortion is the divergence. The divergence is a function and the curl is a vector field.

Did somebody want to ask a question now? I'm not going to use the shape distortion. In dimension two, if you do this, you have a rotation, so you have only one parameter, so the curl is a function. The volume distortion is still the divergence. The shape distortion is the Beltrami coefficient, which we're not going to talk about.

[So the first one here is determinant, what is the second?]

You can decompose a matrix as PO, where P is positive definite and O is orthogonal. If I do this to first order then I get $1 + p + o + \dots$, where p is symmetric and o is skew-symmetric. Exponentiate this to get the relation.

So the curl is this rotation of space produced by the vector field. The divergence is the distortion of volume infinitessimally. There's some homology you can do with the shape distortion too, but in a more elaborate context. This is like World War II but we're still in the 19th century. Then you have a theorem, that if you have a region and a vector field, maybe you have a cut and you have a vector field across a boundary. You ask how much material crosses the boundary. So you have a piece tangent to the boundary and get $\int_{\delta R} N_V = \int_R div\ V$. I think that's called Gauss' theorem, I don't remember. Does anybody know? That's Gauss' theorem?

Now we do the third one. This is the most interesting one. Say you have a vector field and a closed curve; at every point you look at the tangent of the vector field along the curve. You look at $\int_\gamma t_v(\gamma)$ and it's the circulation of V around γ. If you have any surface bounded by γ then this is equal to $\int_S N\ curl\ V$.

If you have a (repeated) vector field in the plane, then the curl points out of the plane, and call it the vorticity of V. I think this is called Green's theorem, right? I'm not sure. I never liked it because it had a big formula that I don't understand.

To compute the curl you need some partials and so on.

I don't know exactly who figured this out; it may have been Poincaré. See, all of these things, let's see, there are some identities now. All these things use the fact that I'm in Euclidean space. I'm using the metric to get the normal and I'm using the volume for the integral.

One of the nice things is to make this covariant under diffeomorphisms changing coordinates. This calculus is attributed to Cartan, but he said he was just writing down what Poincaré was doing.

So invariant language. We have

$$functions \to^{grad} vector\ fields \to^{curl} vector\ fields \to^{div} functions.$$

We could also do

$$functions \to^{grad} vectorfields \to^{div} functions.$$

So is this 0? This is $\partial_x^1 + \partial_y^2 + \partial_z^3$. So the div of the grad is $\partial_{xx} + \partial_{yy} + \partial_{zz}$. This is the Laplacian, so this is not zero; this is essentially an isomorphism; it has finite dimensional kernel and cokernel. This is not part of the invariant language.

But each of the other two compositions is 0. The curl of the grad is 0, and the div of the curl is 0. The other thing you can do is curl curl. This is like a Laplacian on vector fields, and is again not 0. You can also do div grad which is nonzero. In the invariant language we're going to define new concepts. We're going to make our diagram

$$functions \to^d 1 - forms \to^d 2 - forms \to^d 3 - forms,$$

and these relations are $d^2 = 0$. In dimension two this would be

$$functions \to^d 1 - forms \to^d 2 - forms$$

and then it stops. Here the curl is a function again. You have a theorem of Euler that rotation in 3-space has an axis.

So these are, you can say in the spirit, the functions at the end are not really functions, they're volume forms. You can multiply volume forms by functions and get other volume forms, so there's a bijection, and the functions at the end are volume forms in disguise.

The definition of a 1-form is a field of linear functions on tangent vectors. You can integrate this along a path. If we call our elements f, η, ω, Ω, then we have $\int_\gamma \eta$, where we can parameterize a path γ however we want to. Then you show that this integral is well-defined.

This makes sense anywhere you have tangent vectors, so you can make an induced function on tangent vectors. See, vector fields can only be transformed by isomorphisms. The 1-forms have total invariance, but in one dimension, and they can be integrated.

A two-form $\omega(x, y)$ is a skew-symmetric bilinear function on pairs of tangent vectors.

A three-form is a skew-symmetric function on triples, linear in each variable. I'll say what d is, but the three integration theorems become special cases of Stoke's theorem, which says $\int_{\delta R} \star = \int_R d\star$. See, $\star$ can be a function and R can be one-dimensional, so that you're integrating the gradient over the path, and the integral over the boundary is just the sum.

We're up to about 1850, but Stoke's theorem was written in the other language, not in this language.

You can integrate a one-form over a path, a 2-form over a surface, and a 3-form over a volume. Then you have this one relation. We have three such equations, and I discussed the three equations. Without knowing what d is, yet, if you imagine that we're working with a 1-form, and you can take a little square, and integrate around the boundary and get a number. If the square has sides ϵ then the trivial bound is that this has order ϵ. But if it's reasonably smooth things start to cancel and you get order ϵ^2. Then you compute d and the integral is practically a constant depending on the area. Whatever the operator is, it's basically determined by the integral equation of d. Now, $\delta\delta(R) = 0$. So if we define d this way, you have to have $d^2 = 0$. What Poincaré did was work out what d is in many definitions.

To define $d(\star)$, you take a small rectangle and take the integral of $\star$ around the boundary and look at the ϵ^2 term, the leading term, and you divide by ϵ^2. So $d(\star) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \int_{\delta R \equiv \epsilon} 1 - form$.

So the Poincaré lemma is $d \cdot d = 0$. If this formula is going to be true then $d \cdot d$ had better be 0.

So the way I like to define d, is that functions eat (x, y, z). Then one forms are $X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz$. At a point the X, Y, Z give you constants. Then a two-form looks like $X(x, y, z)dydz + \cdots$, and a three-form is $\phi(x, y, z)dxdydz$. So $df = f_x dx + f_y dy + f_z dz$. So $df(v)$ is the directional derivative of f in the direction of v. This is the one we already understand.

At my history at the IGS, the institute in France, I kept finding myself explaining to physicists the difference between *df* and *grad f*. If you don't see the difference between the vector field, which requires the Riemannian metric, and the 1-form, which is invariant, then you should think about it.

So you have a little algebra, the exterior algebra, and then the space of forms forms a graded commutative algebra. Then d is defined like this on functions and then extended to the whole algebra to be a derivation. You want

$$d(\eta \wedge \eta') = d\eta \wedge \eta' + (-1)^{|\eta|}\eta \wedge d\eta'.$$

So this formula will give you a unique extension. $d(Xdx + Y dy + Zdz) = dX \wedge dx + dY \wedge dy + dZ \wedge dz$. Similarly, you get for a two-form $d(Xdydz + \cdots) = dX \wedge dy \wedge dz + \cdots = (X_x + Y_y + Z_z)dxdydz$. So this map is secretly the divergence. The first map is secretly the gradient; then the second, where you have cancellationa and get six terms, if you write it out you get the curl.

This is the calculus of differential forms. It's a graded commutative algebra with an operator which squares to 0. The k-dimensional elements can be integrated over k-dimensional manifolds.

Poincaré had all of this, the formulas for d and so on. So questions? I'm going pretty fast. If you want to slow down a little bit, the original cast of characters, functions, vector fields, and then div, grad, and curl, well, functions can be thought of as functions or volume forms, and vector fields can be thought of as 1-forms or 2-forms, and the three operations are given the same name d.

This is a different way to talk about div, grad, and curl, except now we cannot form the Laplacian, $grad \circ div$. The Laplacian requires the identification of 1-forms and 2-forms, which requires a metric. So it's important to pay attention to the things which do not depend on the metric.

What can I say in four minutes? I mean, this is slightly clarified from the way Poincaré was doing, he was using coordinates and he knew how to change coordinates. But, Poincaré was working 50 years after Riemann, Maxwell. Something happened in between. Riemann studied integrals of 1-forms along paths in surfaces, Abelian functions, the genus was important. He found there were $2g$ independent integrals. Riemann was talking about $\phi(z)dz$, one-forms in complex variables. Such things are automatically closed. He knew that if $\eta = \phi(z)dz$, then $\int_{\gamma_1} \eta = \int_{\gamma_2} \eta$, when γ_1, γ_2, are homologous. This can be interpreted in terms of Stoke's theorem as $\int_R d\eta = 0$.

So basically what Poincaré did, was looking at Riemann said, hey, let's do this in all dimensions. Algebraic topology was born under the integral sign of Stoke's theorem. Riemann, down in Italy, worked with Betti, who studied multidimensional integrals in higher dimensions.

This is a 1-dimensional integral and 0-dimensional boundary.