Algebraic Topology
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Gabriel C. Drummond-Cole

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So let me just review quickly. What we've done is defined H_1 and H_2 of any space $\mathbb{X}$. For H_1 we consider equivalence classes of closed directed curves mapping to $\mathbb{X}$ or closed oriented surfaces mapping to $\mathbb{X}$ for H_2 up to deformation.

I don't think I correctly said what deformation is. So here we have two closed curves mapping into $\mathbb{X}$ and here we should have a cylinder mapping into $\mathbb{X}$ so that the boundaries agree with the original two maps.

For surfaces, it's the same. I'm being a little pedantic here. You put the first surface on one end and the other surface on the other end and connect them with a cylinder and map it into the space in the same way.

For homotopy you'd map a space cross an interval; these differ subtly; it's a logical point. In a deformation one could be red and the other blue.

This also allows surgeries, like when a vertical smoothing becomes a horizontal smoothing in H_1.

In H_2 you have two kinds of move. If in some place the surface touches itself you can connect that with a cylinder. Or, in the inverse, if a cylinder can be shrunk to a point, you can seperate the ends. In the first case the inverse is equal to itself. You come in to the intersection and change directions. If you do this twice you get back to the first case. In the second case it's not. One move increases the genus, the other decreases.

So this is the definition of homology in dimensions one and two, and then this just defines a set, but these are actually Abelian groups so that you can define, well, there are two ways that I can, uh, yeah, let's, I want this to be connected, well, I guess I'll make a remark.

Notice that if $\mathbb{X}$ is path-connected, and if we extended this discussion to disconnected collections of closed curve and closed surfaces and we considered disconnected objects up to deformations and these local moves, then each equivalence class contains connected representatives.

You can do surgery between different components no matter their various orientations. But you get the same set if you allow disconnected subobjects, as long as the ambient space is path connected. In this extended discussion disjoint union of domains and maps into $\mathbb{X}$ defines a binary operation $+$.

So if you have something mapping into $\mathbb{X}$ you take two copies, coloring one domain blue and another red. This is that logical point.

Logically the sets $A \amalg B$ and $B \amalg A$ are different but there's an equivalence so that they're equivalent up to deformation. You take $A \amalg B \times I$ and then cross them at the last minute. If you take a lot of things, it doesn't matter how you arrange them on the table. This is why I had to take deformation instead of homotopy.

There's a little funny thing that if you have two parallel planes then you don't quite want to just connect them with a tube. You have to do a flip, like a little bit of a Klein bottle.

This may seem tedious but the point is we started with the geometry and now this happens when you get to algebra. If we started with algebra this would all be trivial, but then you wouldn't have a picture.

So now you have to check that $+$ is well-defined on equivalence classes. Just think your way through that. And now I claim that the collection of equivalence classes is a set. But before I put the relation on it wasn't a set. I don't want to talk about this point for too long.

There's one object in mathematics which I know of that doesn't have to form a set, that's a category; the collection of objects doesn't have to be a set.

But once we have an equivalence relation we can get a bijection with a set, like the set of continuous maps from the standard circle into our space.

Now we have a binary operation on our set and we can ask if it's a group. You can see that the trivial map of the circle in is like an additive 0. Similarly, the 2-sphere mapping in is an additive identity in dimension 2. So there's an additive identity. The argument with interchanging factors show it's commutative. Inverses are obtained by looking at the map from the object cross itself, with one inside the other.

So let me just start that proof.

Exercise 1 *Show that a surface with one orientation and the surface with the opposite orientation are additive inverses.*

Proposition 1 *So we get this from the exercise, that H_1, H_2 are abelian groups.*

Last time I gave an argument that H_1 of an oriented closed surface is isomorphic to $\mathbb{Z}^{2g}$. The idea was to look at the intersection with the first cut with the Riemann cutting system.

You have a curve and you look at the intersection number with the first Riemann cut. Every time it touches the first cut it crosses it transversally. You introduce a standard curve, add its negative, and then the union has intersection number 0. So you can pull the union off and then do the same thing, look at the intersection number with the next Riemann cut. So you add multiples of standard curves until you can pull your curve off. So we get an intersection number with each cut. There are $2g$-cuts.

The way I said it last time is you have an intersection number with the first cut which gives you a homomorphism to $\mathbb{Z}$. This gives you a kernel and you look at the map from that kernel to $\mathbb{Z}$, so you kind of deduce this algebraically. But if you do these cuts nicely you can just read off the invariants. I didn't want to get to technical, but we have total control of the geometric object.

This object is not always a direct sum of copies of $\mathbb{Z}$. If you take the M obius strib and cut it down the center, it will go twice around. That curve going twice around, in the space $\mathbb{X}$, the M obius strip. So going twice around is the same as two copies; going around twice is the same as going around the boundary. Now close this manifold by sewing on a disk. You get something homeomorphic to the real projective plane, lines in three space.

Exercise 2 *Show that the real projective plane is homeomorphic to the disc with a m obius strip sewn on.*

Then the circle through the band is of order 2 or 0. How do you know that it's nonzero? You can get an intersection number in $\mathbb{Z}_2$, though not $\mathbb{Z}$. But the center line intersects one of our cuts one time, so it's not 0. So $H_1(\mathbb{RP}_2)$ is $\mathbb{Z}_2$.

[What do you interpret this as?]

This is the same H_1. I'm just computing it. I'm going a little fast. If you're on a surface, even if nonorientable, you can look at the intersection number modulo two. First it's invariant, and then it's a homomorphism.

One reason it's an invariant, under deformation first, is that if you start to move it all you can do locally is pull something across with a plus and a minus.

Slightly more formally. Let $\mathbb{X}$ be the surface and γ the curve. You can intersect the cylinder of the deformation, and then the intersection points are boundary elements of curves and have to pair up, since every one manifold has an even number of boundary components.

This class is nonzero, it was part of the first paper on topology proper, by Poincaré.

We'll see more examples before we get an abstract statement.

Say you have a cycle moving in. By general position you can find a point it doesn't cover. Then you can push the circle out to the M obius strip, so it has to go around the circle some number of times. So every curve in the M obius strip is a multiple of the center line, but twice the center line is 0, but it's nonzero. So we know it's $\mathbb{Z}_2$.

Poincaré duality tells you that this is always the way such an example arises. That's the meaning of Poincaré duality.

The standard curve was also nonzero; since there are only two elements they are homologous. In projective geometry, this corresponds to the idea that a point in the plane is a line. Any two planes intersect in a line in space in $\mathbb{R}^3$, so similarly every pair of lines meets in a point.

The fact that you have Poincaré duality is a lot like the axiom of projective geometry, that any two lines intersect in a point.

So now I wanted to mention, were there any questions about this?

[Digression on projective geometry, and other and sundry geometries]

I want to take a couple of examples, with $\mathbb{X}$ as a 3-manifold or a 4-manifold. So let it be a 3-manifold. It could be $S^3 = \mathbb{R}^3 \cup \infty$, or $S^1 \times S^2$, or $S^1 \times S^1 \times S^1$.

So the first thing to observe is that if you just drew at random, it would be unlikely that you'd ever cross your path, so any path can be moved slightly to make an embedding. So every homology class is generated by embedded circles. You can't do this in two dimensions if you only use deformations. With surgery you can seperate in dimension 2. In higher dimensions you can make everything embedded. Now with this geometric picture, you could say "I like these embedded things." You can look at this picture and require that you keep things embeddings. This is called isotopy, and gives knot theory, which is complicated.

Look at a small sphere. There are certain numbers of spaces where the knot is entering and leaving, and you might imagine doing reconnectings, and introducing relation. These are called skein relations and this is a field of current research. The talk Wednesday by Justin Sawon was about this kind of things, indirectly.

Whereas homology can be said algebraically, these things can't be said algebraically yet. There is discussion but it's not understood. Mathematicians are like guys riding big wild horses. They've managed to get on top of the horses, but we're not in control. Whereas since Steenrod, we're in control of homology. Anyway, this is called quantum theory.

In dimension four if you do this there's nothing to say because you can draw it with no crossings. Isotopy and deformation are there the same. There's no additional discussion in dimension 4. If you consider families of curves, then 1-dimensional families have interesting intersections, but you're moving off into other directions. So it's only going into dimension three that you get this rich theory.

Proposition 2 *Up to homology equivalence you can make a* 2*-manifold in a* 3*-manifold or* 4*-manifold embedded.*

Think of the surface as a graph attached to an n-gon. I can pretend the graph is embedded since it's one dimensional. Now the surface will move around and cut through itself. You perform a surgery-like operation along the intersection, and then the vertices go down to the graph and hit the graph somewhere. You get a 2-sphere containing a bunch of closed curves. Look at them in the plane, and then we can put igloos on the closed curves and span them by disjoint surfaces. Your original surface have this as boundary, and so do your igloos. Put the surface together with a ball, so the change in the ball is homologous.

Each element of $H_2(M_3)$ is represented by a surface of genus g. You can get an interesting norm from this on the second homology group, called the Thurston norm. How do you prove it in a 4-manifold? You have a map into a 4-manifold. It's a graph with a cell attached, and you make the cell embedded. The intersection of two surfaces in 4-space is generically in points.

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