Algebraic Topology
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I'm going to give a geometric definition of the low dimensional homology groups of a space. Later I'll give an algebraic definition in all dimensions. The geometric definition gets more difficult in higher dimensions. Later still we'll show that these two definitions are equivalent when both are defined. I wanted to give another example first.

Last time we talked about if you have a surface, possibly nonorientable, then you define for each closed curve a number which is either 0 or 1 depending on whether a neighborhood of it is twisted or not. This gives a map from closed curves to $\{0,1\}$, and we extended this to unions of closed curves by linearity. We showed that if you did an action of switching smoothings at an intersection, that didn't change the number.

I wanted to give another example of such a homomorphism. Take a fixed closed curve γ. We'll call this $\cdot\gamma$. Count the number of intersections of a curve (in general position) with γ and reduce it modulo 2. The same smoothing operation can be done away from γ.

The first homomorphism was invariant under deformation, and so is this one. Are there any questions about these examples?

This intersection with γ is interesting. You can use it in the following way. Say I have a complicated curve. You can move two adjacent curves crossing the curve close together and then smooth these off the curve altogether. If there are an even number of such curves you then get a system of closed curves in the surface cut open along γ. We can define a set of invariants, which are intersecting with the curves γ.

Eventually we'll move the thing into our simply connected region, if all these things are zero. I'll add a relation that says a curve in a small neighborhood of a point is 0. Maybe this already follows, since it is equal to two times itself. So you don't need to add that relation.

There were $2g$ cuts and so that gives you $2g$ invariants. If the coefficients for each of these is 0, then you get 0. This is kind of sneaking up on the algebraic definition.

[How do you do it if there is a 1?]

You're supposed to be looking at a cutting system. You can look at these in essentially any order. So you compute the intersection with any of the cuts, in any order. Maybe we should just stick to the special case where they're all zero. A later thereom will be that if you define these as invariants there is an isomorphism. This is Poincaré duality. I think you should be able to understand that the intersection with γ is defined.

Did I give you a problem to guess what the different equivalence classes are? What's your name?

[Yasha]

Yasha? What Yasha's suggesting is that you take the curve X. You compute the invariant for γ_1. If it's zero, then you repeat as above. If not then you just add a standard curve S_1, and $X + S_1$ can be moved off. So we'll get $X = \sum a_i S_i$ where the S_i consist of a bunch of standard things. So there are 2^{2g} equivalence classes.

[how do you know that these exist?]

I've told you that there are fifteen or twenty arguments in math; this is one of them. The cut was chosen to be nonseparating, so two close points can be connected one to the other.

This is like choosing a basis. I don't like it, but the end justifies the means.

[Can you do this with $\gamma_j \cap S_i = \delta_{ij}$? Do you care?]

I think I can, but I don't care.

[lost?]

Every equivalence class contains a connected representative. So this is kind of what homology theory is about. This works in the nonorientable case, but it can be $2g + 1$. But, this works in the, this is kind of the natural thing to do in the nonorientable case. In the orientable case there's something better to do, so let's do it.

Now let's have the orientable discussion. Now we consider closed directed curves on the surface. We can take a fixed γ again, this time directed, and consider curves up to deformation and up to oriented reconnection. In other words, if you have two branches that are close together, you have to smooth them in an orientation preserving way. So when you cross x to γ you check the agreement of the orientations of the crossing arc. Now you can define an intersection number in $\mathbb{Z}$. If you deform the curve, it will change by an even number of terms, with opposite signs. So up to oriented reconnection and deformation, this number doesn't change.

Now you can do the same argument. You look along γ, so that if they add up to 0, you can locally seperate your curve off of γ. This is because somewhere you have adjacent plus and minus signs. So now you get $2g$ integers.

So that's homology theory, there's nothing else except words now, except the objects get more complicated and you don't have Poincaré duality, which gives you a way to compute, which is nice.

So what do I mean by a space? It depends on the audience. Some of you have not had the definition of a topological space, who's willing to admit it. It doesn't help that much, though. Any space with a metric has a topology. Cell complexes are included in this. Systems of equations have an induced topology from affine or projective space. Functions have a topology. For this definition, you don't need it to be connected, but this will have no connection between components.

Definition 1 *The first homology group $H_1(\mathbb{X}, \mathbb{Z})$ of a connected space $\mathbb{X}$ is the set of equivalence classes of finite unions of closed directed curves in $\mathbb{X}$ modulo deformation and oriented reconnection. The rank of this group is called the first Betti number.*

An oriented closed curve is a map of S^1 into the space.

This is geometry so the words aren't supposed to be elegant, the pictures are supposed to be elegant.

So union gives us $+$ on the set. Since we're allowing disconnected pieces we can take union. Up to equivalence we can bring a loop of one over near the other one and connect it up. So every equivalence class is represented by a connected curve. Deformation means that you move the map by a parameter t so that you have a map of a cylinder into the space. This is also called free homotopy.

[free because there are no points fixed?]

Yeah, usually there are fixed points so sometimes you say free to emphasize this. So that's the definition.

Of course, algebraically, you can do H_1 over any commutative ring. I'm not sure about that. I have to think about that.

I claim, or we've sketched a proof, that this is isomorphic to $(\mathbb{Z})^{2g}$. We see this because the i factor in the product corresponds to the intersection number with γ_i.

So this will follow from the later combinatorial definition; if you have a reasonably compact space, then this group is always a finitely generated abelian group, so it's $\mathbb{Z}^d G$ where G is a finite abelian group.

[You'd have to show that these $2g$ cuts are linearly independent. But that would be proving that the things were at least that big.]

There's a sort of filtration every time you cut. You intersect with the first cut, which gives a homomorphism to $\mathbb{Z}$. Then the kernel is the curves that can be moved off. Now there's a homomorphism to $\mathbb{Z}$ which is the intersection with the second cut. These are always onto. This gets to 0 after $2g$ steps.

I should do this more systematically. I didn't think I was going to be calculating this today, but I just couldn't help myself.

So Riemann discovered this group, and gave it an analytic interpretation. He made up an elaborate theory of θ functions, which is still being studied.

What time is it? 1:50? So twenty minutes?

There is another interpretation of this equivalence relation. Suppose you start with a curve in a space, moving through an equivalent curve. If I move it through a homotopy this will look like a cylinder. Then I perform a surgery to get to two loops. Then there is an action by mapping in a pair of pants.

Let's say that a little more carefully. Look at the levels of the pair of pants. They begin with single circles, there is a singularity with a one-point intersection of two circles, and then there are two disjoint circles.

The language is that you have two cycles which are equivalent, with the actual equivalence given by this surface.

Another way to define the equivalence relation is to draw the domain. You have two cycles and then an arbitrary surface connecting between them and a map of the whole thing into the space. Then you think of a projection and look to the levels. So the other way of saying it is that two curves are homologous if there is a surface with the two cycles as boundary in the range space. With cylinders this would be free homotopy.

So when we go to dimension two we want to find a two-cycle. A two-cycle is a two-dimensional homology with trivial boundary. Suppose you could make your boundary components tiny and then fill them up. So a two-cycle is a map of an oriented closed surface into $\mathbb{X}$.

Definition 2 *The second homology group $H_2(\mathbb{X}, \mathbb{Z})$ is the set of equivalence classes of 2D-cycles in $\mathbb{X}$ up to deformation and surgery. The rank of this group is called the second Betti number*

Have you heard of surgery? In the sixties it was a big thing, but it exhausted itself by its own success.

So one example of surgery is to cut along a closed curve and cap off with disks. So you shrink down (if you can) to a point and squeeze off the ends.

[Isn't everything equivalent to a sphere?]

No, although in the simply connected case we are; then this is the Hurewitz theorem.

[Are these orientable?]

Yeah, this is orientable. A cycle is a map of an orientable surface into $\mathbb{X}$.

You can also do the opposite surgery. Is that all? Yeah, that's all you can do. There are a couple of trivial ones. If you have a 2-sphere in a neighborhood of a point, that's zero. This follows from the logic of what we've said.

So that's the geometric picture of $2D$-homology.

The fact that the algebraic definition agrees with the geometric definition is not something that will be in a textbook. It was completely known in the 10s and 20s, but it was forgotten as algebraic methods took over.

How long? It's been twenty minutes? No?

So now we can, I had a way, well, let's do a simple example. Look at $S^1 \times S^2$. You take a 2-sphere and cross it with an interval, so you get a spherical shell, and you identify the inner and outer shells. In this space, you could pick a 2-sphere between the two shells. It turns out that H_1 and H_2 are both $\mathbb{Z}$.

There's a famous theorem of Poincaré that in a 3-manifold, the first and second Betti numbers are equal. Later we'll find a first and third homology group, which for three manifolds will be $\mathbb{Z}$.

[What is the equivalent idea of what you said for one dimension in the two case?]

If you continue with this pattern you'll start to deviate from the algebraic definition. But here you define a homology to be a 3-manifold with two 2-cycles as boundaries. You'll have only two types of critical point, and those will be the connection and disconnection. Look at $x^2 + y^2 - z^2$. That looks like two cones, but then they come together into a cylinder or move apart into hyperboloids.

Well, I'm trying to do Morse theory in one minute. You can do this in dimension n and end up with the wrong definition of homology. Around dimension four or five this starts to deviate. I'm using this geometric definition in low dimensions.

This is a little forboding because I don't know what all 3-manifolds are. In the algebraic story you have kernels and images of del. Now the objects have to be, if you interpret them geometrically you don't quite get this. We'll discuss this later.

Some of you have heard of bordism and K-theory, they can be derived from this wrong definition. That's too much to say now, it's not very important. We should have a good geometric understanding of what first and second homology is. Most of mathematics use the first and second homology. Every now and then there's a little bit of H_3, and in the general theory you need all of them, but when you're looking at elements, like a symplectic structure somehow corresponds to an element of H_2.