Algebraic Topology
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Let's recall Stokes' theorem again. If we were to break the definition of homology into two steps, well, right now let's work in an open set U in Euclidean space. We have differential forms on U, and given one on U we want to solve on U the equation $\omega = d\eta$.

The first condition is $d\omega = 0$. Given this, then for any Z which is a linear combination of domains of integration in U so that the boundary of Z is zero, you get a second condition, that $\int_Z \omega = \int_Z d\eta = \int_{\delta Z} \eta = 0$. So the integral of ω over a cycle is zero.

The first condition is the derivative condition and the second condition is the integral condition. Furthermore, if Z_1, Z_2 are two cycles and $Z_1 - Z_2 = \delta W$ then $\int_{Z_1} \omega - \int_{Z_2} \omega = \int_{Z_1 - Z_2} \omega = \int_{\delta W} \omega = \int_W d\omega = 0$. So a closed form ω has "periods" $\int_K \omega$, where K is a boundary class of cycles, $Z_1 \sim Z_2$ if $Z_1 - Z_2 = \delta W$ and all of these must vanish if $\omega = d\eta$.

More generatlly, if $\omega_1 \sim \omega_2$ when $\omega_1 - \omega_2 = d\eta_{12}$ then such an equivalence class has periods on homology classes of cycles.

The point is that given Stokes' theorem, just the formal properties arise naturally in trying to solve this equation. So Betti started looking at these homology classes of cycles, and he defined the Betti numbers, in terms of the maximum p such that for any collection of $p+1$ cycles there is a relation $\delta W = \sum z_i$ for some subset of indices.

I don't know what else he did since this was in Italian.

Then Poincaré made a slightly different definition. He noticed that sometimes a multiple of a cycle bounded something. He actually builds these interesting manifolds by gluing together sides of a cube. So if $4z = \delta W$ then $4 \int_Z \omega = \int_{4Z} \omega = \int_{\delta W} \omega = \int_W d\omega = 0$. So the Betti number is the number of generators but the Poincaré Betti number ignores torsion.

Then you can extend this to manifolds; if you have a diffeomorphism from U_1 to U_2 you can glue along the diffeomorphism and still get something with a differentiable structure.

You can talk about closed manifolds, so that you get something without boundary, and about oriented manifolds.

The Betti numbers are symmetric so that the k and $n-k$ Betti numbers are the same. Heegard gave objections and came up with a counterexample. For example, $\mathbb{RP}^3$ homology is $\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}$. For the second definition you lose torsion and get $1, 0, 0, 1$. Poincaré even explained that Heegard's example was actually mentioned in his first paper. Then Poincaré pointed out that his proof seemed to work for both definitions so it wasn't right. So he gave the proof again in the second paper.

I think the proof in the first paper was largely right. If an $n-1$ manifold seperates an n manifold then it's homologous to zero; if it doesn't then you can connect two points on opposite sides of the $n-1$ manifold with a path in the complement. One can actually keep going in the argument and get the second dimensional case. That proved it up to five.

The way Poincaré proved this was to make a third definition, of the induced Betti numbers. There's some good geometry in this. He assumed the manifold was a union of these domains of integration. Then you restrict this discussion to cycles and homologies made out of only these pieces. He reduced the number of pieces used. This is the way we think about homology now, with simplicial complexes or CW complexes. Then he showed that these were invariant under subdivision.

As to the question of whether this is a combinatorial manifold, he studied things made out of cones. He only really went through dimension three. Then he talked about the case where all the pieces were simply connected, which meant that they were cells. The reduced Betti numbers then turned out to be the same as regular Betti numbers.

Then using these nice cells he formed the dual decomposition. The point is that for each cell in the original, there was a cell of complementaty dimension in the dual complex. Then the incidence matrix under the duality bijection, whose ij entry asks if i is a face of j, is the transpose.

I bet he was scared for a little while. He had to come up with a bunch of new ideas. This is kind of a sketch of a bunch of ideas, which have been discussed a lot.

So this is basically based on the idea of having weighted subsets. If we think about cycles and homology, there are two viewpoints. One is that they are subsets with geometric structure. The other viewpoint is that they are maps of abstract versions into a space. These two points of view are not the same, because a map can crush things. You have to do some work to relate them. Each has its advantages for developing different properties here.

Right now what Scott's doing is sometimes he wants to be in one of these, sometimes in another. There's an interesting thing going on right now in quantum physics related to this. So Gromov-Witten theory is defined in terms of maps of abstract things in. And there's a conjecture by physicists, well, these give you a rational number but the physicists have a conjecture that these rational numbers, summed in a certain way, $\sum \frac{gw_n^{ij}}{h^z - n}$ are integers. Anyone in here studying symplectic topology? You wouldn't be in here, you wouldn't know you should be in here if you were already studying symplectic topology. The problem, the tension is just between these two viewpoints in ordinary homology theory.

So this last discussion was really about subsets, about moving subsets around. I wanted to break it up into two parts, a subset part and a mapping part. Let's go back over the definition of homology. Part one would be for open subsets of Euclidean space and various generalizations of piecwise manifolds.

Just imagine taking unions of domains of integration; you can get many smooth manifolds this way and also things with singularities. Whenever you glue together they're always like manifolds with corners.

Here define cycles and homologies with integer coefficients by linear combinations of domains of integration.

There's another point I want to make, that for differential forms you don't really have to have a manifold. When you glue pieces together, like if you have three things coming together, and suppose you're considering 1-forms. You only care about integrating them along paths. So a 1-form here is defined everywhere except along the singularity. You just ask that three one-forms agree on the vector where they're all defined, i.e., the one where you stay on the discontinuity. This is another point, that differential forms have a natural definition on manifolds with corners. The dual objects are cones on things that aren't spheres.

So if you have nice objects you can work with subset-defined homology but then when I had my first course in algebraic topology, Steenrod was my professor and he explained how to extend homology to all spaces. Say you have an arbitrary space. Well, let me not take the above as a definition, just as a first step on nice spaces.

A cycle is going to be a map from a cycle in a nice space into the arbitrary space. As a subset it can't be viewed as a cycle. This is Steenrod's version of singular homology. I only looked at books that were written in 1964, so I don't know if it's written in any book. The one that's written in most books is Eilenberg's definition, where you talk about mapping the standard simplex into your arbitrary space, and take a homology on that. This is good for algebra and not so good for geometry, Steenrod's is good for geometry and not so good for algebra.

The difference is that you get a fixed object as your domain in Eilenberg's definition, whereas for Steenrod you get any closed manifold. The Steenrod thing right now is a proper class.

Since I'm giving a class in the geometric viewpoint I'm going to erase the Eilenberg definition. What I heard is that they were planning on writing up everything, and they didn't write up the second book because of mathematical differences.

If you have two cycles then there should be a big space and a homology between the two is an embedded homology including an embedding of the ambient space. Now you're not just mapping the cycle, but the whole carrier as well.

Steenrod's definition has no chain complex.

These singular definitions are functorial; there's a natural transformation. The subset definition is not obviously functorial. This definition came around World War II. Poincaré's definition was older, but the question arose, do homeomorphic spaces have a common subdivision?

So one reason to return to this geometric picture is that, even though the Eilenberg definition is clean and fast, it's much more connected to the older viewpoint of cells. The interesting invariants that were discovered since the 1980s are not topological invariants, they're combinatorial. Topology as the study of invariance under homeomorphism is not correct anymore. Now there are more invariants in the combinatorial or geometric structure. It's good to learn homology from this viewpoint which is closer to the combinatorial way. Eilenberg only uses the topological structure of the space, well, so does the other, but it has more.