Algebraic Topology
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Gabriel C. Drummond-Cole

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If f is a self-homeomorphism of an orientable surface Σ of genus $g > 1$ there is a $p \leq 2g - 2$ and a $q \in \Sigma$ such that $f^p(q) = q$. This is due to Nielson.

We're aiming to get to a point where we can prove most of this. We'll prove it for $g > 2$.

This number is sharp. For surfaces with boundary, the question of the smallest integer is still open, although Moira Chas' thesis was about this.

We'll also show that if $f : K \to K$, a map of finite polyhedra, induces isomorphism in homology and no sequence is periodic then the Euler characteristic is 0.

It's formulaicly true for $g = 1$, but irrelevant. This we'll prove. Let me go back over some things from last time. So I'm going to go a little fast today and then Scott is going to amplify maybe.

Given a cell complex X define a chain complex by $\cdots \to C_{i+1} \to C_i \to \cdots \to C_0 \to 0$, where C_i is the free abelian group generated by $(cell, orientation)$ with the relation $(cell, orientation) + (cell, -orientation) = 0$. This is isomorphic to taking the free abelian group generated by the cells. If there's one cell, you take two $\mathbb{Z}'$ and identify $(-x, 0)$ with $(0, x)$. This is the way the professionals do it, and it makes it more natural. You get a boundary map where you take the sum of the boundary cells with the induced orientations.

If you go to a codimension two face, go to the boundary and then go to the boundary again the codimension two faces appear twice with opposite orientations.

Define $B_i \subset C_i$ as the image of $\delta : C_{i+1} \to C_i$ so $C_{i+1} \to B_i \to 0$ and $Z_i \subset C_i$ as the kernel of $\delta : C_i \to C_{i-1}$ so $0 \to Z_i \to C_i \to B_{i-1}$. Then define $H_i = Z_i/B_i$ so $0 \to B_i \subset Z_i \to H_i \to 0$. These are the basic short exact sequences.

Let me define subdivision. If you have a cell complex, you choose a sphere in the boundary of some cell, and then add the cone on that sphere. That's the basic subdivision operation. Iterating this, well, if you choose an empty, well, a cell is something that is a cone on a sphere. After doing one subdivision you get C'_* which I claim is isomorphic to $C_* \oplus \mathbb{Z}$. I claim as well that taking a direct sum of $\mathbb{Z}$ with two adjacent terms does not affect homology and that in fact if two complexes have the same homology they differ by a sequence of such moves.

Start with a big cell Y and divide it into $y + y'$. So you take C'_* and embed C_* into it by writing Y as $y + y'$. Then every other generator is a generator without subdividing. The other generator is the new boundary face x. Then if I just try to put y and x in it sort of works, modulo error terms.

I thought I could just write this down. Write $Y = y + y'$ to map C_* into C'_*. That's a chain map. This is an embedding. Look at the quotient complex. It's exactly $\mathbb{Z} \cong \mathbb{Z}$. That's because the boundary of y is x plus something in C_*. Symbolically $\delta y = x + C_*$. So we get $0 \to C_* \to C'_* \to (\mathbb{Z} \cong \mathbb{Z}) \to 0$. So you can lift this back and get a direct sum decomposition.

[What does it mean for chain complexes to be isomorphic]

There's always a meaningful incumbent definition of isomorphism when a mathematician makes a definition. If you have another chain complex you have a bunch of groups and a bunch of maps so that the diagrams commute.

So the summands can be y and δy. Let's form this complex. Formally add a generator $\tilde{y}$ and another generator $\delta\tilde{y}$ with boundary $\tilde{y} \to \delta\tilde{y} \to 0$. Then map this sum to C'_* by taking $\tilde{y}, \delta\tilde{y}$ to $y, \delta y$, and Y maps to $y + y'$, everything else mapping by the identity map.

So we'll just take this to be the definition of subdivision.

It's clear that homology is not going to change because you have two things whose homologies don't talk to each other. Homology respects the biproduct (direct sum). Scott on Friday will be responsible for all my sins today, because I'll be away in Florida. I won't be back until Tuesday.

Now let's specialize to simplicial complexes. Then we have chain complexes. A simplicial complex is a cell complex so we can apply this machinery. Suppose you have a simplicial map $f_\Delta : K \to L$. This gives you a chain map $f_\#$. This is $f_\#(\sigma, or) = -f_\#(\sigma, -or)$ if nondegenerate, 0 otherwise.

Proposition 1 $\delta f_\# = f_\# \delta$.

As a little aside, this is what Gabriel is going to talk about. Suppose the simplicial complex has dimension d and the d-simplices can be oriented so that the sum is a cycle, and this is connected. Call this a (geometric) cycle. Suppose K and L are geometric cycles and suppose f is a simplicial map which is nondegenerate in dimension d by an orientation preserving map. This is the definition of a branched covering, a very interesting class of mappings. It looks like a covering except you can wind around codimension two loci. There is a rich supply of such maps that Gabriel will show you. For example, holomorphic maps with a certain property are always branched coverings.

So we want to show that $\delta f_\# - f_\# \delta(\sigma) = 0$. If σ is nondegenerate then faces are mapped nondegenerately. This is called a transported structure in mathematics. You're moving everything over by a map. In the degenerate case suppose this collapses two vertices, like this. There's a little nice thing that happens. One term is zero because of degeneracy. Two boundaries are mapped nondegenerately so that they cancel. This is always true. When you collapse a tetrahedron, draw one on the blackboard. Here's a map of the tetrahedron into the plane. You take a balloon and collapse it to a sheet. You'll get two copies, a front and a back. A corollary of this is that $f_\# : C_i(K) \to C_i(L)$ induces maps $B_i(K) \to B_i(L)$ and $Z_i(K) \to Z_i(L)$ so induces $f_* : H_i(K) \to H_i(L)$. This is the celebrated induced transformation for a simplicial map.

Now we have the simplicial approximation theorem. Suppose $f : K \to L$ is continuous, and that f takes a simplex into a star of a simplex of L. We can always achieve this if K, L are finite. Actually you just need K to be finite. The stars are an open cover which has a Lebesgue number. You can always achieve this by subdividing K enough. Then there is an f_t preserving $*$ with $f_0 = f, f_1 = f_\Delta$ a simplicial map of $K' \to L$ where this is the first barycentric subdivision. Here f_t takes a point to a segment of a star.

I'll be more precise in a second. This is proved by induction, [proof by pictures]. If two stars intersect nontrivially, then their intersection is somehow related to a star. This isn't the proof I had in mind.

We want to find a homotopy; you choose a point and everything is a cone on this. This will be mapped simplicially.

As we subdivide nothing changes in homology. The fact that we've moved the map by a homotopy means that the induced map on homology is the same. This is another, so,

Proposition 2 *To a continuous f we get a homology f_*.*

Deform f by f_t to f_Δ consider $f_\#, f_*$ associated to simplicial maps.

Theorem 1 *Lefschetz fixed point theorem.*
Let $f : K \to K$ (finite polyhedron) with $fx \neq x$ for all x. Then

$$trace(f_* H_{odd}) = trace(f_* H_{even}).$$

We say $Lf = \sum(-1)^i trace f_*(H(i, Q)) = 0$.

If you take a very fine triangulation, the image of a simplex might be very long. You take a subdivision finer than the minimum distance from x to fx. Then subdivide for the star condition. Then use a homotopy to make this simplicial, so that you still don't get any fixed points. Now remember that this is subdivided, so that a little triangle actually goes to the sum of more little triangles. Then the chain map has the property that every basis element maps to a bunch of basis elements far away. So now you get zeros on the diagonal. You'll actually get more zeros near the diagonal if you index this correctly. So you deduce that the trace of $f_\#$ is 0. Then you use your exact sequences: $0 \to Z_i \to C_i \to B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$. So we get that the trace of C_i will be the sum of the traces of B_{i-1}

and Z_i, which is then the sum of the traces of B_{i-1}, B_i, and H_i. Now take alternating sums and get that the alternating sum of the traces of H_i is zero.

We can apply the theorem to every iterate of f to prove the Euler characteristic 0 thing. Let A be the matrix of f on H in even homology and B the matrix of f on H in odd homology. Then $tr(A) = tr(B)$ and functorially (I should have said, this was the first functor in mathematics) $tr(A^2) = tr(B^2)$. Now, also, these are invertible since we induced an isomorphism on homology.

If you have automorphisms of two vector spaces and all of the traces of the powers are zero then the dimensions of the spaces are equal.

Are you familiar with this lemma? The trace is like the sum of the eigenvalues, the trace squared is like an order two symmetric polynomial, and so on. On the other hand A has a characteristic polynomial whose coefficients are the elementary symmetric functions.

We have the same for B. The traces are equal and then we have this formula connecting s_i (powers) and c_i (elementary symmetric functions). The c_i generate the powers; with rational coefficients the s_i generate the c_i. I don't exactly remember the formula. There's a nice way to write it but I don't remember. Well, there's some nice formula. Going the other way you only need integer coefficients. So what we know is that the s for A and B are all equal so the c for A and B are also equal. These are the coefficients for the characteristic polynomial. We also know by invertibility that the constant coefficient is nonzero. So the lists have the same length because the other ones are zero, above the dimension.

It seems obvious now but I don't know how to finish. You have more work, Scott.

These formulas are called Newton's formulas.

There's a cool formula, something like $\frac{1}{\det 1-tA} = \sum \frac{tr(A^n)t^{n-1}}{n}$. That might not be exact right. These are called zeta functions.

Let me say just one more sentence. If you have a homeomorphism, and the homology is Q in degrees 0 and 2 and you know that the size of the matrix is $2g$, then the trace of the matrix in homology, well, the trace will be two. All of the traces of the powers will be 2. If you raise that to a high enough power you get a contradiction. It's written in Moira's thesis. You only need to look at one page. It uses the fact that the matrix is a symplectic matrix too. I'll discuss this.

One philosophy is that the homology functor gets you into algebra.