Algebraic Topology
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Okay, uh,

Theorem 1 *let f be a continuous mapping between two simplicial complexes $K \to^f L$ where L has the property star-star. Then there exists a simplicial subdivision of K called K'' and a simplicial mapping $K'' \to^{f''} L$ which is homotopic to f.*
Moreover, the homotopy may be taken to be as small as desired by subdividing enough (possibly in the range as well.

[Is this what you were indicating last time by crumpling up the paper and stomping on it?]

No, that was an example of a map. This is saying that you can make such a map simplicial by subdividing and perturbing slightly.

Let me define terms.

Definition 1 *A (finite) simplicial complex is a finite set together with a collection of subsets (called k simplices if the cardinality is $k + 1$) closed under taking subsets and containing all singletons*

A simplicial mapping sends vertices (0-simplices) to vertices and subsets to subsets, linearly on each simplex in these subsets.

You can use the existence of real numbers to make this a geometric simplex by considering a simplex on the vertices $(1, \cdots, n)$ to be the collection of points $(\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$. Here you can see α_1 as the volume on the simplex attached to the face opposite 1, and so on.

Simplicial dimension can never go up in such a map (by definition). If it projects down some number of dimensions it's called degenerate.

Next week Gabriel is going to talk to us about simplicial maps which are nondegenerate.

[On the left board you say you can make the homotopy as small as desired. What does that mean?]

You can put a metric on a simplicial complex, and then it's the maximum distance between f and f''.

[Do you need to subdivide L as well?]

I don't think so. We'll see.

[This is the "just start proving it" technique.]

I organize mathematics by arguments, not statements. The central point is that the star of each vertex is contractible. That's it. The second idea is that if you have what's called the Lebesgue number of a covering, well, let's have a metric. Make every edge of unit length and then we have a metric. Make each simplex a generalized equilateral triangle.

If you have a compact set then for a covering there is a δ any set of radius δ sits entirely in one element of the covering.

The key ideas of the proofs are, again, that the stars of the vertices are cones, and therefore contractible, the Lebesgue number of a covering, and induction.

1. Open stars of vertices in L cover L and have a Lebesgue number. The star of something means all the larger things that contain it. When you have a star you have a nerver which is the actual complex.

   Let's hope I don't have to use stars of stars, I remember something about that thirty years ago.

   If we want to do things small you subdivide so the stars are small. If I stand at Huntington or the back of the room or whatever, I don't want to be able to see the stars.

2. By continuity of f for some subdivision of K, called K' the image of each simplex of K' has diameter smaller than the Lebesgue number of the set of stars of vertices, which cover L.

   Make it real tiny so the images are not too big. I have to keep doing this at every stage.

3. Suppose we have constructed a homotopy of f with desired properties on the $k-1$ skeleton of K'.

   I'm going to do barycentric subdivision and get even more points. Oh, it looks like I need stars of stars as well. Let me just say the inductive step and go back.

   The map is simplicial so the map is simplicial on this subcomplex and goes where the vertices go. The rest of the map is going around in here. So all I do is add another vertex in the center and map that to the center, and it extends linearly. Now I have two maps of this simplex which agree on the boundary.

> Star-star means that stars of stars are contractible, so maps of the sphere can be extended to maps of the ball. You start inductively and then whatever extension you have there.
>
> I'm using two things: that L has contractible stars of stars of vertices, the star of the closure of the first star. Stars are closed sets, and I'll say open stars when I mean the interiors.
>
> The union of the opposite faces don't touch the vertex. They're called the link.
>
> So I'm using that L has contractible star stars. For example, if you take a tetrahedron it doesn't have the property. The boundary of a tetrahedron, I mean.
>
> But if you subdivide a little bit you're okay.

The idea is that you haven't covered much ground taking the star of the star. I'm paying a little bit now because I didn't use that book.

I didn't want to say homotopy because it's technical, and deformation is more casual, but I'm using the homotopy extension principle, which says that if you build a homotopy on a subcomplex then you can extend it to the entire map.

[Can you make that more precise?]

A homotopy of a map can be extended. I was greatly helped by a picture in Milnor's notes about isotopy. The picture here is so dumb it doesn't stick with you.

Isotopy makes you keep everything a smooth homeomorphism. The molecules move by homeomorphism. You can make every molecule in the room move.

[This is what you use for knot equivalence?]

Yes. Here's a subset that you're moving with time. If this moves along like this, you need some reasonable assumptions, but if this is a submanifold you can think of the velocity vector field whose vertical component is one and whose horizontal component tells you how you're moving in space. Then by an elementary lemma you can extend this vector field to the whole manifold. You extend it in a collar and let it damp out to the vertical field, then you can integrate it to get a motion, which is the motion you want.

Anyway, that's isotopy extension; this is homotopy extension. You push down by induction. You need this to be a full subcomplex, so that it contains subsimplices.

The proof is always induction on simplices.

So in this inductive proof we'll homotope the map, then extend the homotopy. We have it defined on the skeleton and then extend it to any old homotopy. I actually have to have a precise statement.

For me the more crucial step is to look at the new map defined on the skeleton that is simplicial. That'll go into, well, there's a little more to say.

The picture is that this triangle has two maps on it. There's the simplicial map and the original map homotoped to be correct on a skeleton, so it's just sort of going around nearby. You always have to be careful you're in a contractible part of the space.

I have to give you a better precise statement to give you a formal proof. When you extend these homotopies you're working locally. Even if we had many stars of stars I could subdivide enough to make this okay.

I don't want to think now because I have an appointment in New London Connecticut at five o'clock and I have to drive around because there's no ferry. I'll let you question me about details next Monday.

The first application is that we can deform a Steenrod cycle for homology to a Poincaré cycle.

A Steenrod cycle is an abstract cycle mapping into a simplicial complex. I can subdivide and then the map in deforms it to a simplicial map. If you look at the top dimensional simplices if they map nondegenerately, then that gives you a cycle in the subdivision. Similarly if you have homology between the two you subdivide and homotope it and that gives a homology between the images. There's a relative version where you can leave it alone on a subcomplex.

In terms of the original one we've now subdivided and now all these pieces are mapping simplicially. If you take the preimage of this this is the thing I was talking about. Now when you make it cellular it's in good form for making that count.

So this leads to the induced transformation on Poincaré homology. This is the second application. In fact it leads to the induced chain mapping. If you have a map between two cell complexes you can approximate the map by a simplicial map and then look at the nondegenerate part of the map and then any chain maps to the image. A simplex in K maps to 0 or a K-simplex. In a nondegenerate map each oriented simplex maps to an orientation. That's where you use nondegeneracy. Then you extend to linear combinations of oriented simplices. So you get $f_\# : C_iK \to C_iL$. This commutes with the boundary. Then $\delta f_\# = f_\# \delta$. So you get an induced map on homology $f_* : H_iK \to H_iL$. This is consistent with the obvious one given on the Steenrod homology.

There are a couple of short exact sequences that are useful.

You have $0 \to Z_i \hookrightarrow C \to B_{i-1} \to 0$, where Z is the kernel (cycles) and B is the image, boundaries, and $0 \to B_i \to Z_i \to H_i \to 0$.

Then the Lefschetz theorem says $f : K \to K$ has no fixed points then after some subdivisions you get a matrix with no diagonal, so trace zero. Then the alternating sum $\sum(-1)^i tr\ f^i_\#$ So then you get $tr(C_i) = tr(Z_i) + tr(B_{i-1})$ so the alternating sum is the same as $\sum(-1)^i(tr(Z_i) - tr(B_i)) = \sum(-1)^i tr\ f^i_*$. This is the famous Lefschetz formula.