Algebraic Topology
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Here's a cycle and a homology with a lot of vertices, just cellular objects, and then you suppose you have the projection map, and every vertex just goes down, and you get a linear ordering, cycle one going in and cycle two going out.

I want some too, what is that? I see food and I want to eat it. And then, so you subdivide so that this is a simplicial complex, and extend it linearly over complexes. So the preimage of a point on this tetrahedron is a triangle square, or triangle. So combinatorially between vertices you never change your type. Every edge maps nondegenerately. So we're studying this with something like a Morse function so that as you, well, I want the vertices at the ends to be separated as well, I want all the vertices to be separated.

Scott, your remark is throwing me off.

[I was concerned that if you map an interval to a point that may affect your preimage.]

There's another local move that you need, I was going to do it later. We're trying to describe homology out of other operations. One is deformation. Combinatorially nothing is changing.

I confused the cycle with the carrier of the cycle. Logically the cycle is the chain on the carrier.

1. deformation
2. mapping cylinder homology

   The idea is suppose you have the circle mapping around twice. Then take the cycle in the image, two copies of the circle and there's something called the mapping cylinder which connects x to $f(x)$ by an interval. For $f : X \to Y$ it's $I \times X \cup Y/(x, 1) \sim f(x)$. So this gives a cylinder coming down onto the circle. It's a homology between going around twice and two copies of going around once.

   So now if you look at a fiber at a point within the homology, you have some vertices to the right and some to the left. If you move it back there's a collapsing map, a mapping cylinder. There's another one at the other side.

I used those a lot without saying it.

3. Local moves.

   So when we move if we don't cross a vertex everything moves the same way. When I cross a vertex all the lines move across except at that vertex, so let's blow that up a little bit. You have some lines coming in and some lines going out. So then if I look at this space I have all of these lines, and I have everything, if I look at, if I call the link of this point the opposite face of every simplex that touches this vertex. The vertices are divided into two parts, some crushed to a point and some to which the point explodes. Look at the full subcomplex generated by the vertices here.

   Some simplices will be entirely on one or the other side; some will cross the singularity. So you have three kinds of simplices. The whole neighborhood of this is the cone on all of that.

   So a more realistic picture of this, well, a pretty realistic picture is the tetrahedron itself. Take the boundary of this tetrahedron. These are the two vertices to the left, these are the two vertices to the right, and then the rest of this boundary runs across.

   So in the surface of the tetrahedron you get a little surface bounding these edges. And the boundary is being moved across by a combinatorial isomorphism. So the boundaries of the two neighborhoods are isomorphic. So you have two mapping cylinders with something coming across.

   These things complete to cycles. These "vertices" I've drawn are not the original vertices, but the shadows of them.

   So let me make the statement. You go to the vertex, you look at the vertices collapsed to a point, and you look at the full subcomplex that contains them, then you take a neighborhood of it in this thing, and the same thing on the other side.

   I'm drawing these things like manifolds because the whole thing is a cycle in the middle, so a manifold with singularities. So the boundary here is a manifold with singularities. The statement is that for the cycle to cross the singularity, outside a neighborhood it just moves by a product. But on the neighborhood, you cut out a complex and glue in another with the same boundary. So you cross the vertex by a local surgery.

So you might have a point and the subcomplex might be two pieces, then the subcomplex on the other side might look like an annulus, these are the neighborhoods of the subcomplexes. Then things go smoothly across the point, so that the outer and inner boundaries may be smoothly connected to the boundaries of the circles. Then the cone on that thing is what happens near the point.

Okay, so I'm not saying it very cleanly, but the essential point is that this is just determined by the vertices so you only need to look at easy pictures to verify this.

[It seems like the mapping cylinder homology is just a specialization of the cone.]

I tried and I couldn't quite see how to do it. The thing is to show that a mapping cylinder homology can be made out of these. I think that's possible too. Just organize the collapsing bit by bit, so to speak.

Exercise 1 *write two as a composition of 1 and 3.*

Maybe it's like $\frac{1+3}{2}$.

[Laughter.]

Maybe that's kind of right, somehow. I think that should be a pretty easy exercise, I just don't see how to do it right now. Actually, I was thinking about refining this a little more. Because of Morse theory, when this is a manifold, the critical points can't do this complicated surgery. You cut the neighborhood of one sphere and put in a neighborhood of another.

So this is a kind of research problem, what are the irreducible local moves. This gets into the singularities a little more. You want to find the cobordism of $\mathbb{CP}^2$ to zero that has relatively high dimensional singularities, not just a point.

You know when everything's a manifold you can have much more precise local moves. I started thinking about it last night and then I fell asleep. You can't do better than moving across with a product of spheres. So you start by making your homology as much a sphere as you can.

[Yasha: can't that be two spheres?]

No, maybe you could have S^0 becoming S^1, but that's different.

So an annulus is a 2-sphere minus two discs, so you glue in the neighborhood of S^0 and get the 2-sphere, then take the cone on that and get the 3-ball.

[What makes a move atomic?]

You can break the other moves up into compositions of atomic moves.

These moves aren't so bad. You just crush something down and blow it back out to something with the same boundary.

So now you know what homology means.

So we have the Steenrod singular homology of a space. We have cycles carried on something and then maps into a space. Homologies are given by some third space that contains the two and then maps in.

This was not the first way homology was defined. The first way was to work on the carrier. Poincaré's definition was for a space divided into cells. So $\delta(\sigma\alpha_i c_i) = 0$ makes Z_1 a cycle, and $Z_1 \sim Z_2$ if their difference is a boundary of cells.

Theorem 1 *The Poincaré homology of a space divided into cells mapping to Steenrod singular homology by the obvious inclusion is a bijection.*

They didn't know this for a long time because they didn't have the definition. Let's prove this.

Onto and into will be the same argument applied first to cycles and then to homologies.

Let's draw the picture. Here's the cell decomposition. I want to show that the natural map is onto, so I want to take a Steenrod singular cycle mapping into the space.

Well, first I break it up into a lot of little pieces, first looking at the vertices. Forget this statement. Keep the same vertices. Well, the point is, the basic idea is, this is easier, when you look at this one-dimensional thing, then you can have things, well, we don't want the dimension to blow up. Then you just take a point and I move it into the appropriate subskeleton. So if it's a 1-cycle it misses a point in a 2-cell and then I can project it to its boundary. So you can keep doint that until you get down to the right dimension.

Depending on where I put the hole it may go to one or the other side, so you get blown out to something of the right dimension. So this is a deformation. Now I want to look at this thing and pick out the cycles that it's supposed to correspond to. It's already working here, but notice that the vertices are all mapping to the vertices. But when you're doing a 2-dimensional one, you'll already have done this to it. Then the faces go into the 2-skeleton. Then you have to develop more theory, but I'll give you the idea. You have this abstract cycle, which is of dimension two. I've arranged the vertices to go to the vertices. What we know is we have this mapping into the one-skeleton somehow.

The idea is to take a point in the interior, so that this is a cellular map. I need to redo that. After deforming it will be moving around somehow in nice pieces. After some subdivision each piece will map nicely. Let me pretend it's nice, otherwise it's dead.

[He crumples up a piece of paper and stomps on it.]

It'll be like this, a nice map, folding and crushing, and so on. We'll do that next time. Suppose you made it reasonably nice, what is the reduced cycle it corresponds to? It's pretty easy. Intuitively it goes around like this, and then you cancel according to orientation. There's a sign attached according to direction.

That's going to be the Poincaré cycle you want, you have to prove it's homologous. You have to drill little holes and then stretch this out and homotope the map and get a new map where each disk maps onto the whole thing. Then all the other stuff will be crushed out to the boundary. That'll be part of the proof. We'll manipulate this thing.

You can see how there's work involved here, it's not completely trivial.

One corollary of this, well,

Corollary 1 *Poincaré homology is independent of the cell decomposition.*

Poincaré only knew this for subdivision, but Milnor found a polyhedron with two decompositions with no common subdivision.

Corollary 2 *Homology is functorial. If you have a map between two spaces you get a map induced on the homology groups. That is $f : X \to Y$ induces $f_* : H_*(X) \to H_*(Y)$ which* *respects composition and identity.*

The functor was born at this moment, with this proof.

Corollary 3 *Poincaré homology is homotopy invariant, a homotopy functor.*

Using the Steenrod definition, if you have a homotopy between X and Y then it's obvious that they have the same homology, just using deformations. Then two spaces with homotopy equivalence induce isomorphisms at the level of homology.

There's a lot of nonfunctorial homology too.

[What's the dimension with the failure of subdivision?]

Five, six, or seven. No. Sorry, four. There are algebraic varieties, nice complex manifolds that are homeomorphic but not combinatorially equivalent. There were five-manifolds found in the sixties. Milnor's work was roughly a cone on a lens space cross $\mathbb{R}$ or $\mathbb{R}^2$.

I'll tell you next time, let me think. Historically there are three sets of counterexamples. You can have infinitely many smooth algebraic varieties that are homeomorphic but not combinatorially equivalent.