Algebraic Topology
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Excuse me for being late. I need a little more chalk if you can find some.

I want to work an exercise that he gave me, first he distracts me and then he doesn't show up.

Let's describe local moves to generate all manifolds. I haven't defined what a manifold is, but we can let this be the definition, all things you can get from these.

The idea is Morse theory, and the picture is to think of a function on a manifold, and then if you look at the levels, you have things like spheres being born, saddles, and so on.

In the picture I've drawn there are ten moments. The first and last are kind of inverse from eachother; you start or complete a ball. There's kind of an elementary result that you don't need any intermediate minima or maxima. It's kind of an elementary cancellation lemma, suppose your function only has one minimum and one maximum.

So what are the moves to go from a sphere to a sphere? There are actually two statements here. Every manifold of dimension $n-1$ can be obtained from the sphere in this way, or this track is every manifold of dimension n. That's not right, sorry. Every manifold is obtained like this, and if there is a manifold between two manifolds, then you can get from one to the other by these moves.

So the statements are that every closed n-manifold is a path of handle moves from S^{n-1} to itself, union two n-discs and, more generally, if two manifolds are cobordant then you can get one from the other by these moves; we say that one is obtained from the other by local surgeries.

Here's the basic move: you add a handle to move from one sphere to two. A handle is $D^p \times D^q$ and the boundary of it is $\delta D^p \times D^q \cup D^p \times \delta D^q$. The lemma says that you can choose p, q positive except for the first and last time.

In dimension three, there's a lot of battling going on between the $1, 2$ and the $2, 1$ cases. You can do a bunch of $1, 2$s and then get rid of them with $2, 1$s. The weakness of this is that it's not unique. In dimension two this is fairly easy to study, but it is still not unique there.

So you can think of these as paths in the space of manifolds. The move in dimension n is very simple; you just stick on a disc along half its boundary. In the smooth category you worry a bit about rounding the corners. If we're working with piecewise smooth there's no problem.

If I have a piece of boundary, then there's a collar neighborhood so you can get these moves. The proof is, take a manifold and then just choose a Morse function on it. Then it only has critical points of index $1, 3, 2, 2, 3, 1$, if, say, it's a four-manifold. Between critical points it moves by diffeomorphism. You look in a Morse theory book and that gives you your proof.

You can look at it in dimension n for Morse theory. In dimension $n - 1$, let's look a little more carefully. If you have a sequence $M_1, \ldots, M_n$ of m-manifolds. The procedure to get from one to the next is to cut $S^{p-1} \times D^q$ out of it. You find an embedded sphere whose normal bundle looks like a product.

Has Kevin arrived yet? No.

This contains $S^{p-1} \times S^{q-1}$, and so that's also the boundary of $D^p \times S^{q-1}$, so you can glue that in.

If you do a $2, 2$ move in the 4-manifold, you can glue $S^1 \times D^2$ to $D^2 \times S^1$. You squeeze the solid torus down and then expand it in the other direction.

In fact there's a little theorem, actually, you can get a three manifold from the 3-sphere by doing these surgeries simultaneously.

So we can define, using this, suppose we define, in more innocent times, before World War I, we might have tried to define homology like this. A cycle is just going to be a map of a closed manifold of the same dimension into the space. We say two are homologous if you can put a closed manifold with boundary between them and send it in.

Poincaré had his things embedded, and in codimension one and two, if the target space is a manifold, that's okay. So let's call this bordism homology.

[The problem is that the manifold in between might not exist?]

If we added singularities, if you add cones of things then we get an ordinary homology.

This will be a perfectly good homology with a slight generalization. It turns out this will satify all the axioms except the groups of a point are not correct. What are the groups of a point? Call this Ω and compute Ω_i of a point. The real information has to do with the cycles, which are the bordism groups of closed oriented manifolds. They're generated by closed manifolds with orientation; disjoint union is the sum, and the groups are $\mathbb{Z}, 0, 0, \mathbb{Z}, \mathbb{Z}/2, \ldots$ Usually you get just $\mathbb{Z}, 0, \ldots$ Thom found these. He figured it out rationally and at the prime two; these are torsion except for dimension $4n$. He didn't do it directly; Steenrod asked the question, let's say we take the definition with mapping cycles in, where they may not be manifolds. He asked which could be represented by manifolds. Such are said to be Steenrod representable.

At the same time there was a question about algebraic varieties, another Fields medal, in Japan he's as famous as Tom Cruise here, well, famous, picture on billboards.

In his thesis at Harvard he showed you can resolve a singularity on a variety. Ten years earlier, Thom wasn't sure this was true. Anyway, if you could do it, then, an algebraic variety with singularities is still a cycle. It carries a fundamental class. If you can resolve singularities then it's Steenrod representable. Thom found obstructions to the topological question, in '53, and thought there might be obstructions to the algebraic question. In '63 it was resolved that there were none. The proof is very hard, and is still open in characteristic p. Anyway, that was Thom studying that question, we saw a procedure; if every manifold bounded a manifold then we could make every cycle Steenrod representable.

So that theory, the original question has been forgotten but the results here have gotten a lot of study. If you're studying a natural geometric question, even if it itself may not be useful, having a methodology is a very good thing.

So, this is kind of like the wrong war in the wrong place at the wrong time; it's very interesting but it's not homology theory. It's a pretty way to represent it.

We can say from these two theorems, we have a corollary.

Corollary 1 *The bordism homology equivalence relation is generated by deformations and local surgeries*

It's a generalized deformation where you are also allowed to add these things. I've been doing it all the time, that's the way I defined two dimensional homology. In higher dimensions you have more choices.

I had a great visitor planned for you and he didn't show up. There's a seminar at the CUNY graduate center tomorrow, the first in a series. The speaker tomorrow was supposed to come today.

So what am I doing now? Have I finished this? Okay, so the exercise Slava gave me was to ask what is the analogue of this corollary for ordinary homology? What is Morse theory for manifolds with singularities?

The homologies between things may not be manifolds, they may have singularities, like things with lower codimension.

No, come in, come in, we were expecting Tom Cruise.

[Laughter]

So Stiefel's thesis, in like 1936, was that every three manifold bounds a four manifold.

But any manifold with odd Euler characteristic is not a boundary, because by gluing two of them together you'd get $0 = 2\chi_W - \chi_{\mathbb{C}}$, so that $\chi_{\mathbb{C}}$ is even. That right side is Poincaré duality. So for nonintegral generators, the $\mathbb{CP}^n$ will generate these groups.

The story is that Hirzebruck was at the IAS, and he gets the letter on Thom's result, and by the afternoon he has his result, and writes it up in a nice book, applying all these things to algebraic varieties.

The only thing we don't understand is why things in dimension six are Steenrod representable.

These are not unique; you can connect sum in funny stuff with $\mathbb{CP}^2$, but the map is onto until dimension seven.

Okay, so let's try to, I want to shoot for a corollary for ordinary homology, deformations and some kind of local moves. Let's ee if we can describe what happens; I've never done this before.

[Following what you did, you can collapse a manifold to a point and blow it back up?]

Yes, exactly, you have to prove that.

The goal is to take a deformation plus a local move (equivalent definition of the homology, which is described by the existence of a homology between two maps).

So we need a precise context to work so we can think of every object as carried by a cell complex. So this means a space which is a union of objects called cells, which are balls of some dimension, glued together on boundaries. After subdivision you can assume each cell is a generalized tetrahedron.

In fact, this language of chain complexes, this is a geometric complex. Sitting on every geometric complex is an algebraic complex where you take the linear combinations of these with a boundary operator. These days only the algebraic definition is widely used.

This is by induction, if you subdivide so that the lower skeleton is a simplicial complex, if you add a point in the middle and cone off to the boundary then you get generalized tetrahedra; it's clearly true in dimension one. So then we have a simplicial complex. Sometimes you have to do it again to get things to be embedded.

Sometimes you work with regular cells, so that they don't touch themselves along their boundary; if you aren't working with those you have to do it a second time, this subdivision.

So I want to do some Morse theory. I can think of my whole thing as just being a simplicial complex. Then what I want to do, let me erase this, I'm just going to do it, rather than motivate it.

So the first thing I want to do is to add collars, so I have a nice collar boundary. These are not quite simplices, they're prisms. Then I'm going to make a function. I have all these vertices, and I can number them from 1 to n. I make a function that maps the vertices at one end to $n + 1$, the ones at the other end to zero, and the others to $1, 2, \ldots$ according to the ordering. Then this extends uniquely, linearly to a map on the complex.

This might be wrong but a lot of it will be right.

Now let's study the preimage. When you have one of these maps and you take the preimage of something, you get nice pieces so that unless you're at a vertex, you have some breathing room for the homeomorphism type of the fiber. Here's a nice one: map a tetrahedron to a line by taking opposite edges to the endpoints; then in the middle you get a square or a rectangle everywhere.

[The preimage of a point is a simplicial complex?]

It's a cell complex.

[As a topological space it has singularities?]

Yeah. It'll pick up the same singularities as the space.

The map is roughly a projection near a boundary. I'm sort of verifying again what I just said; the only place there will be a singularity is at a vertex. Some edges will be going to vertices greater than k and others to less than k.

This is very elementary, there's nothing to this, just some linear algebra. What happens, gosh, well, I'll explain this in more detail, but some of the lines are coming into the point and others are going out. Some are being crushed and some are being created. The complex generated on one side is being crushed and another is blown up.

So everything else is preserved; all that happens is that at each integer a set is crushed to a point and then something else is blown up out of that point.

We'll discuss this more.

0.1 Informal talk by Kevin Costello: Algebraic topology and moduli of Riemann surfaces and topological strings and deformation of complex structures and whatever

We're going to informally start. This is Kevin Costello from Imperial College, London. He's going to give an informal talk and then another couple tomorrow at CUNY. I'm going to ask a leading question. What is string theory, in the context of algebraic topology? If you work on string theory, what do you work on?

I suppose everyone knows what a topological field theory is. We have a category S with objects I finite sets and morphisms are Riemann surfaces with parametrized bands. There is no identity. That doesn't matter because there kind of is on homology. This is a symmetric monoidal category. A CFT is a tensor functor F from this functor to whatever. For example, you could have a Hilbert or vector space valued one. The target also has to be one of these. We'll be approximating spectra with chain complexes. Anyway, what this means, say we're going to vector spaces, is that $F(I) \otimes F(J) \to F(I \amalg J)$.

I don't know what string theory is. I don't know if topological string theory and topological conformal field theory are different. TCFT would have more topological targets. We replace the category with singular chains on moduli spaces. So we have C_*S with the same objects and morphisms defined in the obvious sense, $C_*S(I,J) = C_*(S(I,J))$. AS long as C_* is a tensor functor this gives a category.

A TCFT is a tensor functor so we can get $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$. Up to homotopy it doesn't depend on choices of tensors and $*$. So a TCFT is $F : C_*S \to$complexes up to homotopy. The diagram is:

$$\begin{tikzcd} C_*S \arrow[d, "\cong"'] & \\ P \arrow[r] & complexes \end{tikzcd}$$

There exists a minimal model for this which is unique, though not uniquely unique. This is a differential graded category. The idea of an enriched category is where you say the morphisms are blah. The functor along the bottom has to preserve the differential.

When I say topological string theory, I'm referring to this. So this seems to be what arises geometrically.

[Gromov-Witten theory is not exactly this, since it uses compactified moduli spaces. You're later going to bring this in?]

So a first approximation to CFT is $\phi : \phi(\Sigma) : F(I) \to F(J)$. The first approximation is a topological field theory, where this is independent of the complex structure. $\phi(\Sigma)$, that is, as a function on moduli space, is locally constant, so to go from this kind of thing to a TCFT we replace a locally constant sheaf with a resolution.

This operation is a function on the moduli of Σ, so this can be broken up into components. You can imagine this but with values in some locally constant sheaf in moduli space, that seems to arise as well.

So the jump on $S(I,J)$ we have a sheaf, here a constant sheaf, $Hom(F(I), F(J))$, and we're taking sections of it. Over for a $TCFT$ we replace $Hom(F(I), F(J))$ with its canonical resolution $Hom(F(I), F(J))_{\mathbb{C}} \otimes C^*(S(I,J))$. This concept here is really the correct one, and the first one is a truncated version.

One thing this picture shows, is that if we have $\Sigma \in S(I,J)$ it's a 0-chain so it gives an operation $\Phi(\Sigma) : F(I) \to F(J)$. But if Σ_1, Σ_2 are connected then $\Phi(\Sigma_1), \Phi(\Sigma_2)$ are (chain) homotopic and whatever.

an n chain here is an n-parameter family in moduli space.

I said at the beginning that this functor is defined only up to homotopy. But if it's an exact functor, if we can conncect Σ_1, Σ_2 with a path then this is the boundary because you commute with the differential.

$\Phi(d\alpha) = [d, \phi(\alpha)]$.

We say that we work up to homotopy because we've made an arbitrary choice. I don't think this is really a derived category.

We have this category C_*S, and have picked a free resolution for it, for any differential graded category. You take sequences of morphisms. You compose pairs and split them apart. So you use the free resolution or the minimal model. If you just work with the minimal model you never mention homotopy again.

One thing that this guy shows us is that on homology you have $H_xF(I) \to H_xF(J)$.

I should, people who haven't seen this before, if we have a TFT with the property that $F(I) = V^{\otimes I}$. Suppose this is true. Then V is a Frobenius algebra. Here we have, say, a pair of pants which gives a map $V^{\otimes 2} \to V$ which is associative, by the pictures of two pairs of pants attached waist to leg one, to leg two.

[Are the morphisms surfaces up to isomorphism or surfaces? Do you have a fixed universe? Not all surfaces form a set.]

Is this a logic question, I don't understand. Fix a large dimensional vector space.

Higher genus things can be broken up like this in a TFT. I actually had no idea I was coming here to speak today. What should I talk about?

[You described TFTs as functors from a fixed category to something else; that's also how you describe an operad. This is the Frobenius operad?]

Not the Frobenius operad. If you take the category of disjoint unions with at least three, then you get it commutative. If you take the minimal model, it doesn't matter whether it's C^∞ or E^∞.

So we call this category S_0. Then H_*S_0 has the same objects, finite sets, and $H_*S_0(I, J)$ is the homology space of morphisms of this form. The following theorem is due to Getzler:

Theorem 1 *This is the category for the BV operad.*

If you have a tensor from that category to vector spaces then it is a BV algebra. $F : H_*S_0 \to Vect$ is a tensor functor with $F(I) = V^{\otimes I}$ if and only if BV algebraic structure on V, i.e., it is a commutative algebra with $\Delta : V \to V, \Delta \in H_1(S_0(1,1))$. So $S_0(1,1) \cong DiffS^1 \times DiffS^1 \times \mathbb{R}_? \cong S^1$.

Take any chain and take the rotation, that's the morphism corresponding to the Δ structure. There's a pretty picture which explains this.

Dennis was asking about the compactified stuff, $\bar{M}_0$. Instead of using surfaces we use points, spheres with incoming and outgoing points. We do something similar, to get a Frobenius manifold without the pairing. So $H_*\bar{M}_0$ has tensor functors $F : H_*\bar{M}_0 \to Vect$ with $F(I) = V^{\otimes I}$ the same as {family of algebraic structures on V parameterized by V}. Each point in a formal neighborhood of zero gives an algebra structure.

Look at $\bar{M}_0(n,1)$. This is $n+1$ marked points, n incoming, 1 outgoing. These guys form an operad. This differs from the previous case because this is compactified; these are algebraic curves with nodal singularities.

I write down $\bar{M}_0(3,1)$ which generically has four distinct marked points in a ball. But you can have degenerate points where you get something else, something like two manifolds crossing, each containing a couple of these points. So these guys are compact maniforlds. $\bar{M}_0(n,1)$ is compact with dimension $n-2$.

I'll get a field theory for these guys. For V a vector space, then $\alpha \in H_*(\bar{M}_0(n,1))$ has $\phi(\alpha) : V^{\otimes n} \to V$ with the fundamental class $m_n : V^{\otimes n} \to V$. Then this structure is going to be the same as a family of–

[You'd get homology operations on homology of a thing if you did this with chain complexes?]

Yes

[more]

Look at T^*M. Now look to $CF_*(T^*M)$, Floer chains, then that's quasisomorphic to chains on the loop space, since a complex is quasisomorphic to its homology. There's a very general symplectic picture. If X is symplectic then $CF_*(X)$ should be a TCFT. The first here should correspond to Chas Sullivan stuff. The operations come from Riemann surfaces, right, and you can think of Chas-Sullivan multiplication as similar. For geometric reasons there should be a map modulo some difficult symplectic stuff, $CC_*(Fnk\ X) \to CF_*(X)$. So can I just explain why this is a family of algebras? Say $v \in V, v_1, v_2 \in V$. Then $\Sigma_{n\geq 0} m_{n+2}(v_1, v_2, v, v, \cdots, v)/n!$. Pretend these converge; then for each v these are algebra structures. The statement is that this operad is the operad whose algebras are this guy. This homology is precisely known and is described by this. This is what they satify.

Theorem 2 *This is commutative and associative.*

[How is this related to string topology?]

You have the BV version, S_0, and you have S^1 in it.

$$\begin{tikzcd} A \arrow[r, "\subset"] \arrow[d] & S_0 \arrow[d] \\ pt \arrow[r] & \bar{M}_0 \end{tikzcd}$$

You can quotient by the subcategory $S_0//A \cong \bar{M}_0$ up to homotopy. If the chain level version of string topology existed, then S^1 equivariant homology would have this structure I've just been discussing.

[For each n how many operations do you get?]

It's quite big but they're built up. If $n = 2$ then you get 0. This isn't a BV algebra. The BV algebra is S.