Algebraic Topology
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Gabriel C. Drummond-Cole

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So we have this two stage definition of homology. The first idea, Steenrod's definition, is a cycle on a carrier. Maybe it's an open set in Euclidean space with coefficients, i.e., a linear combination of oriented domains with integral coefficients such that $\delta(M) = 0$.

The second part is as maps into a space X. Then two cycles are homologuous if there exists a third carrier C such that $\delta C = Z_1 - Z_2$. Then the equivalence classes give the homology groups.

Let's give a picture in low dimension. For example, what is a 1-cycle? It is a bunch of one dimensional integrable domains, oriented, with coefficients. We need the amounts going into any vertex to equal the amount going out.

Any questions about that?

Now, you can make a more geometric picture by wiggling each of these a little, making every coefficient one. Do this at every vertex. This becomes a closed 1-manifold. So a 1-cycle is a union of closed curves in the space.

Let's try this with a 2-cycle. We'll analyze a 2-cycle then a 2-dimensional object with boundary.

So we have lots of domains of integration with weights and orientation. We can't have any exposed edges; otherwise the boundary is nonzero. So a 2-cycle is something that has no exposed edges, mapping into X.

Study how it looks along an edge. We have faces with weights coming into the edge. Again you see that for orientation, the number of positive and number of negative oriented edges are equal. Cut it transversally to see the same picture as before.

So one can repeat as before; note that we can change orientations to make all coefficients greater than zero. So the total weight inducing o is equal to the total weight inducing $-o$. We can pull apart; note that there are lots of choices here.

[Can you do it without crossing each other?]

Interesting question. Let's continue. We pair positive and negative sheets to get a union of "surfaces." What does it look like around a vertex? It looks like a cell complex where each edge has two sides, the same vertices and faces. Going around a vertex we pass from sheet to sheet, eventually getting back to where we started. There may be more than one way. Generally it looks like the one point union of surfaces. It's almost a surface; we can pull it apart at these vertices to get a union of closed orientable surfaces.

So each 2-cycle pulls apart to a closed surface.

Let's do 3-cycles (this works to dimension five). Start the same way. Again, all faces with multiplicity add up to zero. Take multiple copies with appropriate orientations so that all coefficients are one. Pair them all. Look at what happens around an edge. Again, each wall has a room around it. Again, we get a cycle (or many) around an edge. Again we can pull the edges apart (the same as we did for the vertex before).

An edge in dimension three is the same as a point in dimension two. In general, codimension two is like a vertex in a surface. So this discussion is constant in codimension, and the argument applies to all dimensions to a certain codimension. So a cycle is a manifold down to the codimension to which we succeed.

At a point in a 3-cycle we have a configuration, whose link could be any orientable surface. So a 3-cycle pulls apart to a 3-manifold with possibly finitely many singular points where the neighborhood of a singular point is a cone on an orientable surface, a priori not necessarily connected. But these can be pulled apart; there is a nice fact here:

Exercise 1 *This object is a* 3*-manifold if and only if* $\chi(\textit{object}) = 0$. *(It is crucial tha the cone is connected).*

Now the only problem is something like the cone on a torus. The cone is contractible, so we do a local move, replacing the cone points with a solid handlebody.

For a 4-cycle you do the same steps to get down to the case of cones on a 3-manifold. Steifle showed that every oriented 3-manifold is the boundary of a four manifold. We do the same construction.

For 5-cycles you get cones on 4-manifolds. Then $\mathbb{CP}^2$ is not a boundary of a 5-manifold, which brings us to combordism theory. If you're ambitious you can get one more dimension, and six also. In dimension seven there is a homology class not the image of a closed manifold (Thom).

Let's go back. Do the same argument to construct a 2-manifold. So every 2-dimensional homology is given by a surface with boundary. This is a proof that the new definitions of H_1, H_2 are equivalent to the old definitions of H_1, H_2. Note they both are the same as bordism in these dimensions.

Take a harmonic function. Local moves are the same as passing through critical points. You see this by cleaning up the Steenrod picture to manifolds as before.

We generally have three definitions.
manifolds with singularities $\leftarrow$ manifolds without singularities $\cong$ local move definition (use a Morse function to define equivalence).

Look at morse functions $x^2+y^2+z^2-t^2, x^2+y^2-z^2-y^2, -x^2-y^2-z^2+t^2$.

Exercise 2 *What is the local move corresponding to passing through the morse singularity* $x^2+y^2-z^2-t^2$*?*

How levels change as you pass through a critical point is called a surgery move. As a hint, $\delta(D^2 \times D^1) = D^2 \times \delta D^1 \cup \delta D^2 \times D^1 = D^2 \times S^0 \cup S^1 \times D^1$.